The Category of Hilbert Spaces as an Orthogonal Category [1]

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Abstract

Nishimura [4] introduced the concept of a manual in an orthogonal category in order to formalize the relationships between partially overlapping Boolean algebras in category theoretical language. Our main objectives here are to briefly discuss the axioms and provide a concrete example of a manual. It is demonstrated that one of the axioms in the definition of an orthogonal category can be derived from the other axioms, hence the number of axioms is reduced. We then give a more general definition of a manual, substituting the subcategory in the definition of Nishimura by any category with a faithful functor. We proceed to show that the category of Hilbert spaces equipped with direct sums provides an example of an orthogonal category, after which an example of a manual in the category of Hilbert spaces is provided and discussed in detail. In the last section we outline the connection between manuals and unitary operators on Hilbert spaces.

1 Introduction

It is known that the projection algebra of self-adjoint operators on a Hilbert space is, in general, not Boolean. However, it is possible to choose a set of operators contained in a complete Boolean algebra [1, ch. 7]. The task of quantum logic and set theory then becomes to study the interactions between such sets.

Classically, each set can be thought of as a *Boolean-valued set*, that is, each set X can be equipped with a function from $X \times X$ to a complete Boolean algebra, measuring to what extent two sets are the same in some sense (alternatively, the 'probability' with which an element of X is in both sets). This generalises to the concept of a *subobject classifier* in a suitable category [1, Appendix]. The idea of Nishimura [4] is to find a construction capturing the analogous structure of projection algebras, or more generally, the structure of partially overlapping

[1] This report is an outcome of an undergraduate research project I did during the summer of 2017. The project was supervised by Chris Heunen (School of Informatics) and I was granted an Undergraduate Research Bursary from London Mathematical Society.

operations as introduced by Foulis and Randall [2]. This leads to the notion of an *empirical set* over a *manual*.

Foulis and Randall [2] introduced the notion of a manual with the aim of formalizing the foundations of all empirical sciences; a manual consist of all possible operations that can be performed within a certain theory. The concept of an operation in their formalism is left undefined on purpose so as to ensure generality of the discussion, it is taken as a starting point, everything else being defined rigorously in terms of operations. However, informally, the operations are assumed to correspond to actual physical procedures. Hence the way to think about a manual is to perceive it as a collection of instructions for performing physical procedures which have a meaningful outcome within the theory in question. For quantum mechanics, a manual therefore consist of all operators on Hilbert space corresponding to physical observables.

The subobject classifier for empirical sets takes its values in a manual, which is not necessarily a Boolean algebra due to its empirical nature; this is to say that the logic of an empirical set is determined by the manual. In particular, for the manual of operators on Hilbert space the logic of an empirical set will be precisely the 'correct' logic of quantum mechanics. In this report, however, we will not get as far as empirical sets, and they are mentioned mainly for completeness of the broader view; for detailed discussion the reader is referred to Nishimura [4].

The main aim of this report is to understand the concept of a manual via concrete example, which will be provided by a Hilbert space together with a certain collection of its subspaces. The next two sections are largely repeating the definitions given by Nishimura [4], although we discuss the axioms in more detail in order to motivate them and provide some intuition. Moreover, we show that one of the axioms is redundant, and give a slightly more general definition of a manual. Namely, we do not require a manual to be a subcategory of an orthogonal category, but rather any category equipped with a faithful functor to the orthogonal category; this is motivated by our example, which becomes more interesting with this definition. A manual as defined by Nishimura [4] is then a special case of a manual as defined here.

Since the symmetries of a Hilbert space are not accounted for by the definition of a manual, including the symmetries could be of interest for future research. Thus the concluding section briefly considers unitary operators on Hilbert spaces and their effects on a manual.

2 Orthogonal Categories

We follow Nishimura [4] in defining an orthogonal category. Our motivating example is the category of Hilbert spaces together with the direct sums.

Let $(\mathfrak{K}, \mathfrak{os})$ be a pair consisting of a category $\mathfrak{K}$ and a class of cocones in $\mathfrak{K}$ indexed by a discrete category, called *orthogonal sum diagrams* and denoted by $\mathfrak{os}$. The vertex of the cocone is called the *orthogonal sum* of the base objects. An *orthogonal category* is a pair $(\mathfrak{K}, \mathfrak{os})$ satisfying the conditions (1) to (9).

(1) The category $\mathfrak{K}$ has an initial object.

Initial objects play the role of the 'additive identity' for orthogonal sums, this is made precise by axiom (6). For this reason such an object is called *trivial*.

(2) For any small family of objects $\{X_\lambda\}_{\lambda\in\Lambda}$ in $\mathfrak{K}$, there exists an orthogonal sum diagram having $\{X\}_{\lambda\in\Lambda}$ as the base. That is, there is an object Y and morphisms $\{f_\lambda\}_{\lambda\in\Lambda}$ in $\mathfrak{K}$ such that the cocone $\{X_\lambda \xrightarrow{f_\lambda} Y\}_{\lambda\in\Lambda}$ is in $\mathfrak{os}$.

(3) If both $\{X_\lambda \xrightarrow{f_\lambda} Y\}_{\lambda\in\Lambda}$ and $\{X_\lambda \xrightarrow{g_\lambda} Z\}_{\lambda\in\Lambda}$ are in $\mathfrak{os}$, then there exists a unique morphism $Y \xrightarrow{h} Z$ in $\mathfrak{K}$ such that $g_\lambda = h \circ f_\lambda$ for each $\lambda \in \Lambda$.

Hence any small collection of objects of $\mathfrak{K}$ has an orthogonal sum; moreover, it turns out that (3) implies that two orthogonal sums of the same summands are isomorphic, we will prove this later. In fact (3) 'almost' turns $\mathfrak{os}$ into the class of coproducts in $\mathfrak{K}$; the universal property is however, restricted to orthogonal sum diagrams rather than being true for all cocones.

(4) Given cocones $\{Y_\lambda \xrightarrow{g_\lambda} Z\}_{\lambda\in\Lambda}$ and $\{X_\delta \xrightarrow{f_\delta} Y\}_{\delta\in\Delta_\lambda}$ $(\lambda \in \Lambda)$ in $\mathfrak{K}$, the composite cocone $\{X_\delta \xrightarrow{g_\lambda\circ f_\delta} Z | \lambda \in \Lambda \text{ and } \delta \in \Delta_\lambda\}$ lies in $\mathfrak{os}$ iff all the cocones $\{Y_\lambda \xrightarrow{g_\lambda} Z\}_{\lambda\in\Lambda}$ and $\{X_\delta \xrightarrow{f_\delta} Y\}_{\delta\in\Delta_\lambda}$ $(\lambda \in \Lambda)$ lie in $\mathfrak{os}$, where the sets Δ_λ are mutually disjoint.

(5) If the cocone $\{X_\delta \xrightarrow{f_\delta} Y | \lambda \in \Lambda \text{ and } \delta \in \Delta_\lambda\}$ lies in $\mathfrak{os}$, then there exist cocones $\{X_\delta \xrightarrow{g_\delta} Z_\lambda\}_{\delta\in\Delta_\lambda}$ $(\lambda \in \Lambda)$ and $\{Z_\lambda \xrightarrow{h_\lambda} Y\}_{\lambda\in\Lambda}$ such that $f_\delta = h_\lambda \circ g_\delta$ for any $\lambda \in \Lambda$ and $\delta \in \Delta_\lambda$, where, again, the sets Δ_λ are mutually disjoint.

These two axioms can be summarised as 'composition' and 'factorisation'. Indeed, (4) states that a composition of two orthogonal sum diagrams is itself an orthogonal sum diagram, and that if a composition of two cocones happens to be an orthogonal sum diagram, then so are the cocones. (5), in turn, says that if the indexing set of an orthogonal sum diagram factorises into disjoint sets, then the orthogonal sum diagram factorises into constituent cocones. Note that by (4) these cocones are also orthogonal sum diagrams.

(6) Given cocones $\{X_\lambda \xrightarrow{f_\lambda} Y\}_{\lambda\in\Lambda}$ and $\{X_\delta \xrightarrow{g_\delta} Y\}_{\delta\in\Delta}$ in $\mathfrak{K}$, if X_δ is trivial for all $\delta \in \Delta$, then $\{X_\lambda \xrightarrow{f_\lambda} Y\}_{\lambda\in\Lambda}$ is in $\mathfrak{os}$ iff $\{X_\lambda \xrightarrow{f_\lambda} Y\}_{\lambda\in\Lambda} \cup \{X_\delta \xrightarrow{g_\delta} Y\}_{\delta\in\Delta}$ is in $\mathfrak{os}$.

(7) Given cocones $\{X_\lambda \xrightarrow{f_\lambda} Y\}_{\lambda\in\Lambda}$ and $\{X_\delta \xrightarrow{g_\delta} Y\}_{\delta\in\Delta}$ in $\mathfrak{K}$, if both $\{X_\lambda \xrightarrow{f_\lambda} Y\}_{\lambda\in\Lambda}$ and $\{X_\lambda \xrightarrow{f_\lambda} Y\}_{\lambda\in\Lambda} \cup \{X_\delta \xrightarrow{g_\delta} Y\}_{\delta\in\Delta}$ are in $\mathfrak{os}$, then X_δ is trivial for all $\delta \in \Delta$.

Thus (6) makes precise the role of the trivial objects, they can be added to an orthogonal sum diagram, or taken away from it, without changing the orthogonal sum. Moreover, (7) states that the trivial objects are the only objects that can be added to an orthogonal sum diagram without changing the orthogonal sum, so the orthogonal sums are *maximal* in this sense.

(8) If $f : X \to Y$ is an isomorphism in $\mathfrak{K}$, then the cocone $\{X \xrightarrow{f} Y\}$ is in $\mathfrak{os}$.

(9) Given an orthogonal sum diagram $\{X_\lambda \xrightarrow{f_\lambda} Y\}_{\lambda\in\Lambda}$, if $f_{\lambda_1} = f_{\lambda_2}$ for some distinct $\lambda_1, \lambda_2 \in \Lambda$ (so that $X_{\lambda_1} = X_{\lambda_2}$), then $X_{\lambda_1} = X_{\lambda_2}$ is trivial.

In (9) we require that all the morphisms in an orthogonal sum diagram are distinct up to the trivial maps. The following converse to (8) gives us a hint what the orthogonal sum diagrams are by considering a degenerate case of only one summand.

Proposition 2.1. *If a cocone* $\{X \xrightarrow{f} Y\}$ *is in* $\mathfrak{os}$*, then* f *is an isomorphism.*

Note that in Nishimura [4] Proposition 2.1 is given as another axiom for orthogonal category, it is, however, derivable from the axioms (3) and (8). For this, we need the following result.

Lemma 2.2. *The orthogonal sums of the same summands are isomorphic.*

Proof. Given cocones $\{X_\lambda \xrightarrow{f_\lambda} Y\}_{\lambda\in\Lambda}$ and $\{X_\lambda \xrightarrow{g_\lambda} Z\}_{\lambda\in\Lambda}$ is $\mathfrak{os}$, we have to show that Y and Z are isomorphic. By (3), there exist unique morphisms $h : Y \to Z$ and $i : Z \to Y$ such that $g_\lambda = h \circ f_\lambda$ and $f_\lambda = i \circ g_\lambda$ for all $\lambda \in \Lambda$. Combining these two we get that $h \circ i : Z \to Z$ is the unique morphism such that $g_\lambda = h \circ i \circ g_\lambda$ for all $\lambda \in \Lambda$, which implies that $h \circ i = \mathrm{id}_Z$. Similarly, we get that $i \circ h = \mathrm{id}_Y$, thus we have found the required isomorphism. □

Proposition 2.1 now follows by first noting that $\{X \xrightarrow{\mathrm{id}_X} X\}$ is in $\mathfrak{os}$ by (8), and then applying Lemma 2.2 to orthogonal sum diagrams $\{X \xrightarrow{f} Y\}$ and $\{X \xrightarrow{\mathrm{id}_X} X\}$.

To conclude the definition of an orthogonal category, we introduce some terminology and notation. For brevity, the category $\mathfrak{K}$ itself will be called an orthogonal category by a slight abuse of language. An orthogonal sum of X_λ's is denoted by $\bigoplus_{\lambda\in\Lambda} X_\lambda$. We call a morphism $f : X \to Y$ an *embedding* if there exists a morphism $g : Z \to Y$ such that $X \xrightarrow{f} Y \xleftarrow{g} Z$ is in $\mathfrak{os}$. Two embeddings $f : X \to Y$ and $g : Z \to Y$ with common codomain are *equivalent* if there exists an isomorphism $h : X \to Z$ in $\mathfrak{K}$ such that $f = g \circ h$.

3 Manuals

We next introduce the notion of a *manual* in an orthogonal category. We present a slightly modified version of the definition given by Nishimura [4]. While in [4] a manual is required to be a subcategory of a given orthogonal category, we merely ask the manual to be equipped with a faithful functor to the orthogonal category, thus the notion of a manual as given by Nishimura is a specific case of the definition presented here. The example motivating this slightly more general definition is a decomposition of a given Hilbert space into its orthogonal subspaces together with a choice of an isometry for each subspace, this will be discussed in detail in Section 6. First, however, we will need some nomenclature to be used in the definition.

Given an orthogonal category $\mathfrak{K}$, let a pair $(\mathfrak{M}, \mathfrak{F})$ denote a small category $\mathfrak{M}$ together with a faithful functor $\mathfrak{F} : \mathfrak{M} \to \mathfrak{K}$. We say that a diagram K of $\mathfrak{K}$ *can be pulled back to* $\mathfrak{M}$ if there exists a diagram M in $\mathfrak{M}$ such that $\mathfrak{F}(M) = K$.

Two objects X and Y in $\mathfrak{M}$ are called $\mathfrak{M}$-*orthogonal*, denoted by $X \perp_{\mathfrak{M}} Y$, if there exists a diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$ in $\mathfrak{M}$ such that $\mathfrak{F}(X \xrightarrow{f} Z \xleftarrow{g} Y)$ is in $\mathfrak{os}$. An initial object X of $\mathfrak{M}$ such that $\mathfrak{F}(X)$ is $\mathfrak{K}$-trivial is called $\mathfrak{M}$-*trivial.*

An object X in $\mathfrak{M}$ is $\mathfrak{M}$-*maximal* if for all Y in $\mathfrak{M}$, $X \perp_{\mathfrak{M}} Y$ implies that Y is $\mathfrak{M}$-trivial. Objects X and Y of $\mathfrak{M}$ are $\mathfrak{M}$-*equivalent*, denoted by $X \simeq_{\mathfrak{M}} Y$, if for all Z in $\mathfrak{M}$, $X \perp_{\mathfrak{M}} Z$ iff $Y \perp_{\mathfrak{M}} Z$.

A cocone $D = \{X_\lambda \xrightarrow{f_\lambda} Y\}_{\lambda \in \Lambda}$ in $\mathfrak{M}$, for which $\mathfrak{F}(D)$ is in $\mathfrak{os}$, is said to be an *orthogonal* $\mathfrak{M}$-*sum diagram* if for every cocone $C = \{X_\lambda \xrightarrow{g_\lambda} Z\}_{\lambda \in \Lambda}$ in $\mathfrak{M}$ with the property that $\mathfrak{F}(C)$ is in $\mathfrak{os}$, the unique morphism $\mathfrak{F}(Y) \xrightarrow{h} \mathfrak{F}(Z)$ of axiom (3) can be pulled back to $\mathfrak{M}$. In this case Y is called an *orthogonal* $\mathfrak{M}$-*sum* of X_λ's and is written as $\bigoplus_{\lambda \in \Lambda}^{\mathfrak{M}} X_\lambda$. For only a few objects, we use such notation as $X \oplus_{\mathfrak{M}} Y$. Analogously to an embedding in $\mathfrak{K}$, an $\mathfrak{M}$-*embedding* is a morphism $f : X \to Y$ for which there exists a morphism $g : Z \to Y$ such that $X \xrightarrow{f} Y \xleftarrow{g} Z$ is an orthogonal $\mathfrak{M}$-sum diagram, in which case we say that X is an $\mathfrak{M}$-*subobject* of Y.

A *manual in* $\mathfrak{K}$ (or a $\mathfrak{K}$-*manual*) is a pair $(\mathfrak{M}, \mathfrak{F})$ satisfying the conditions (10) to (17).

(10) For any pair of objects (X, Y) in $\mathfrak{M}$, there is at most one morphism from X to Y in $\mathfrak{M}$.

(11) There exists at least one object X in $\mathfrak{M}$ such that $\mathfrak{F}(X)$ is $\mathfrak{K}$-trivial.

(12) Every object in $\mathfrak{M}$ with the property as in (11) is $\mathfrak{M}$-trivial.

Next we would like to say that if there is a morphism from X to Y, then X is something like a subspace of Y. The trivial objects are then subspaces of any object. Further, we would like $\mathfrak{M}$-orthogonality to mimic the orthogonality of subspaces.

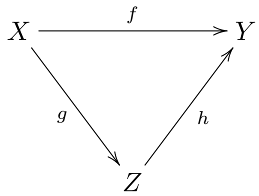
(13) For any objects X and Y in $\mathfrak{M}$, if there is a morphism from X to Y in $\mathfrak{M}$, then for any object Z in $\mathfrak{M}$, $Y \perp_{\mathfrak{M}} Z$ implies $X \perp_{\mathfrak{M}} Z$.

(14) For any $\mathfrak{M}$-orthogonal objects X and Y, there exists an object Z of the form $Z = X \oplus_{\mathfrak{M}} Y$ in $\mathfrak{M}$.

(15) For any object Z of the form $Z = X \oplus_{\mathfrak{M}} Y$ in $\mathfrak{M}$, $X \perp_{\mathfrak{M}} W$ and $Y \perp_{\mathfrak{M}} W$ imply $Z \perp_{\mathfrak{M}} W$ for any object W in $\mathfrak{M}$.

(16) For any objects X and Y in $\mathfrak{M}$, $X \simeq_{\mathfrak{M}} Y$ iff there exists an object Z in $\mathfrak{M}$ such that $X \perp_{\mathfrak{M}} Z$, $Y \perp_{\mathfrak{M}} Z$ and both $X \oplus_{\mathfrak{M}} Z$ and $Y \oplus_{\mathfrak{M}} Z$ are $\mathfrak{M}$-maximal.

(17) For any commutative diagram



of $\mathfrak{K}$, if $f : X \to Y$ and $h : Z \to Y$ can be pulled back to $\mathfrak{M}$ in such a way that the morphism h' in $\mathfrak{M}$, for which $\mathfrak{F}(h') = h$, is an $\mathfrak{M}$-embedding, then $g : X \to Z$ can be pulled back to $\mathfrak{M}$.

A $\mathfrak{K}$-manual $(\mathfrak{M}, \mathfrak{F})$ is *rich* if the following holds:

(18) For any object X in $\mathfrak{M}$ and any embedding $f : Y \to \mathfrak{F}(X)$ in $\mathfrak{K}$, there exists an $\mathfrak{M}$-embedding $f' : Y' \to X$ such that f and $\mathfrak{F}(f')$ are equivalent in $\mathfrak{K}$.

A $\mathfrak{K}$-manual $(\mathfrak{M}, \mathfrak{F})$ is *completely coherent* if the following holds:

(19) For any infinite family $\{X_\lambda\}_{\lambda\in\Lambda}$ of pairwise $\mathfrak{M}$-orthogonal objects in $\mathfrak{M}$, there exists an object Z in $\mathfrak{M}$ such that $Z = \bigoplus^{\mathfrak{M}}_{\lambda\in\Lambda} X_\lambda$.

4 Bounded Coproducts in Hilb

Definition 4.1. *(Bounded linear map, bounded collection of linear maps)* A linear map $f : X \to Y$ is said to be *bounded* if there exists some $M > 0$ such that for all $x \in X$ we have $\|f(x)\|_Y \leq M \|x\|_X$. Let us call the smallest such M M_f. We say that a collection $\{X_i \xrightarrow{f_i} Y\}_{i\in I}$ of bounded linear maps is bounded if the set $\{M_{f_i}\}_{i\in I}$ is bounded.

Let **Hilb** be the category whose objects are all Hilbert spaces and morphisms all bounded linear maps between them. We define the *(external) direct sum* of a family of Hilbert spaces as follows.

Definition 4.2. *(External direct sum of Hilbert spaces)* Let $\{H_i\}_{i\in I}$ be a family of Hilbert spaces. The direct sum of the family is the subset of the Cartesian product of $\{H_i\}_{i\in I}$ containing all those elements $x = (x_i | i \in I)$ for which $\sum_{i\in I} \|x_i\|^2_{H_i} < \infty$. The addition and scalar multiplication are defined termwise, and the inner product of the elements x and y in the direct sum is defined by

$$(x, y) = \sum_{i\in I} (x_i, y_i)_{H_i}.$$

The (external) direct sum as defined above is itself a Hilbert space and is denoted by $\bigoplus_{i\in I} H_i$. Given a Hilbert space H, suppose a collection of its subspaces $\{H_i\}_{i\in I}$ satisfies

1. for $h_i \in H_i$ and $h_j \in H_j$, $(h_i, h_j) = 0$ whenever $i \neq j$, and
2. any element $h \in H$ can be written as $h = \sum_{i\in I} h_i$ such that $\sum_{i\in I} \|h_i\|^2 < \infty$, where $h_i \in H$ for each $i \in I$.

In this case we say that H is the *internal* direct sum of the subspaces $\{H_i\}_{i\in I}$. It is straightforward to see that if H is the internal direct sum of $\{H_i\}_{i\in I}$, then H is isomorphic to the external direct sum $\bigoplus_{i\in I} H_i$; we define the map $f : H \to \bigoplus_{i\in I} H_i$ by $h = \sum_{i\in I} h_i \mapsto x = (h_i | i \in I)$, where each $h_i \in H_i$, and the map $g : \bigoplus_{i\in I} H_i \to H$ by $x = (h_i | i \in I) \mapsto h = \sum_{i\in I} h_i$. Now f and g are bounded linear maps which are mutually inverse. Because of this isomorphism, we will in general not distinguish between internal and external direct sums and will talk about *the* direct sum.

We next define the coprojection morphism for a direct sum.

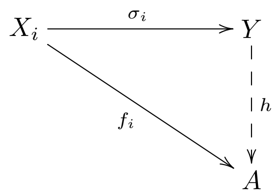
Definition 4.3. *(Coprojection)* Let $\bigoplus_{i\in I} H_i$ be the (external) direct sum of Hilbert spaces $\{H_i\}_{i\in I}$. For each Hilbert space H_λ in the indexed family, define the coprojection $\amalg_\lambda : H_\lambda \to \bigoplus_{i\in I} H_i$ by $v \mapsto (x_i | i \in I)$, where

$$x_i = \begin{cases} v \text{ for } i = \lambda \\ 0 \text{ otherwise.} \end{cases}$$

Observe that each $\amalg_\lambda$ is monic. Further, observe that each $\amalg_\lambda$ has a right inverse sending $x = (x_i | i \in I)$ to x_λ if we restrict its domain to H_λ. It follows that whenever we have $f \circ \amalg_\lambda = g \circ \amalg_\lambda$ for some morphisms f and g and *for all* $\lambda \in \Lambda$, then $f = g$. The maps $\{\amalg_\lambda\}_{\lambda\in\Lambda}$ are said to be *jointly epic.*

Our aim is to find such a collection of cocones in **Hilb** that it becomes an orthogonal category. We note that something like a coproduct structure would do the job, which motivated the introduction of direct sums. However, since we require the linear maps in **Hilb** to be bounded, the direct sums and coprojections do not quite provide the coproduct structure, as demonstrated by Example 4.6. This can be worked around by making the following restriction.

Definition 4.4. *(Bounded coproduct)* A cocone in **Hilb** which is universal on bounded cocones will be called a *bounded coproduct diagram.* Explicitly, a bounded coproduct of a collection of objects $\{X_i\}_{i\in I}$ consists of an object Y together with a bounded collection of morphisms $\{X_i \xrightarrow{\sigma_i} Y\}_{i\in I}$ such that for any *bounded* collection of morphisms $\{X_i \xrightarrow{f_i} A\}_{i\in I}$, there exists a unique morphism $h : Y \to A$ such that the diagram

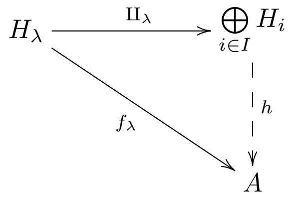


commutes for each $i \in I$.

Note that bounded coproducts satisfy the axiom (3) for orthogonal categories. It thus follows by Lemma 2.2 that bounded coproducts are unique up to a unique isomorphism, we can therefore talk of *the* coproduct of X_i's.

Proposition 4.5. *The bounded coproduct structure of* **Hilb** *is given by the direct sums and the corresponding coprojections.*

Proof. We have to show that given a bounded family of morphisms $f_\lambda : H_\lambda \to A$ $(\lambda \in I)$, there exists a unique morphism $h : \bigoplus_{i\in I} H_i \to A$ such that the diagram



commutes.

For an $x = (x_i | i \in I) \in \bigoplus_{i \in I} H_i$, we define $h(x) = \sum_{i \in I} f_i(x_i)$. Note that h is bounded since we require $\{f_i\}_{i \in I}$ to be bounded. Now clearly $h \circ \amalg_\lambda = f_\lambda$ for each $\lambda \in I$, which proves the existence. For uniqueness, assume there is another such morphism g. We thus have $h \circ \amalg_\lambda = g \circ \amalg_\lambda$ for all $\lambda \in I$, since $\amalg_\lambda$'s are jointly epic, it follows that $g = h$. □

In the construction of the morphism h it is crucial that the collection $\{f_i\}_{i \in I}$ is bounded. We demonstrate this with the following example.

Example 4.6. Let $\{\mathbb{C}_n \xrightarrow{f_n} \mathbb{C}\}_{n \in \mathbb{N}}$ be a set of endomorphisms of the complex numbers $\mathbb{C}_n$ such that each f_n is multiplication by n, that is, $f_n(x) = nx$ for each $n \in \mathbb{N}$. Now each f_n is a bounded linear map, but the set of maps is unbounded. Let $\amalg_\lambda : \mathbb{C}_\lambda \to \bigoplus_{n \in \mathbb{N}} \mathbb{C}_n$ be the corresponding coprojections. If we now try to construct the morphism h as in the proof of Proposition 4.5, for $x = (x_n | n \in \mathbb{N})$ in the direct sum we get $h(x) = \sum_{n \in \mathbb{N}} f_n(x_n) = \sum_{n \in \mathbb{N}} n x_n$, which is not bounded and hence not a morphism in **Hilb**.

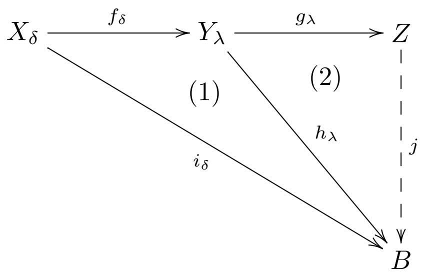
We will denote the class of bounded coproduct diagrams in **Hilb** by $\mathfrak{ds}$. Since coproducts are unique up to an isomorphism, this includes both internal and external direct sums of Hilbert spaces, the maps as defined in 4.3 being one possible representation of general coprojection morphisms.

5 Hilb is Orthogonal

Here we will show that $(\mathbf{Hilb}, \mathfrak{ds})$ is an orthogonal category as defined in Section 2.

(1) is satisfied since **Hilb** has an initial object, namely the zero vector space. The category satisfies axiom (2) since any collection of Hilbert spaces has a direct sum, and the axiom (3) is satisfied by the universal property of bounded coproducts.

To show that (4) holds, first assume that all the diagrams $\{Y_\lambda \xrightarrow{g_\lambda} Z\}_{\lambda \in \Lambda}$ and $\{X_\delta \xrightarrow{f_\delta} Y_\lambda\}_{\delta \in \Delta_\lambda}$ $(\lambda \in \Lambda)$ are in $\mathfrak{ds}$, where the sets Δ_λ are mutually disjoint. We need to show that the diagram $\{X_\delta \xrightarrow{g_\lambda \circ f_\delta} Z | \lambda \in \Lambda \text{ and } \delta \in \Delta_\lambda\}$ is in $\mathfrak{ds}$. To see this, consider the following diagram.



(5.1)

Thus f_δ and g_λ are coprojections, and Y_λ, Z direct sums. Hence for every bounded collection of morphisms $\{X_\delta \xrightarrow{i_\delta} B\}_{\delta \in \Delta_\lambda}$, there is a unique morphism

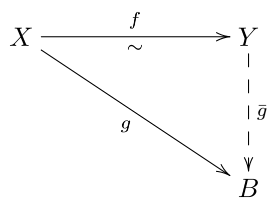
$Y_\lambda \xrightarrow{h_\lambda} B$ such that the triangle (1) commutes. We therefore obtain the bounded collection of morphisms h_λ ($\lambda \in \Lambda$). Again, for this collection there is a unique morphism $Z \xrightarrow{j} B$ such that the triangle (2) commutes. We have thus shown that for every bounded collection of morphisms i_δ ($\lambda \in \Lambda$ and $\delta \in \Delta_\lambda$), there exists a unique morphism j such that the outer triangle commutes. Hence $g_\lambda \circ f_\delta$ is indeed a bounded coproduct and so in $\mathfrak{ds}$.

Conversely, suppose that $\{X_\delta \xrightarrow{g_\lambda \circ f_\delta} Z | \lambda \in \Lambda \text{ and } \delta \in \Delta_\lambda\}$ is a bounded coproduct. We have to show that $\{Y_\lambda \xrightarrow{g_\lambda} Z\}_{\lambda \in \Lambda}$ and $\{X_\delta \xrightarrow{f_\delta} Y_\lambda\}_{\delta \in \Delta_\lambda}$ ($\lambda \in \Lambda$) are bounded coproducts. We begin with the former one. Given any bounded collection of morphisms $\{Y_\lambda \xrightarrow{h_\lambda} B\}_{\lambda \in \Lambda}$, define $i_\delta := h_\lambda \circ f_\delta$ for all $\lambda \in \Lambda$ and $\delta \in \Delta_\lambda$. Now there exists a unique morphism j such that the outer triangle in the diagram 5.1 commutes. Hence we have $j \circ g_\lambda \circ f_\delta = h_\lambda \circ f_\delta$. Since this holds for every $\delta \in \Delta_\lambda$, we must have $j \circ g_\lambda = h_\lambda$ for all $\lambda \in \Lambda$ proving that $\{Y_\lambda \xrightarrow{g_\lambda} Z\}_{\lambda \in \Lambda}$ is indeed a bounded coproduct. Similarly, given any bounded collection $\{X_\delta \xrightarrow{i_\delta} B\}$, there exists a unique j such that the outer triangle in the diagram 5.1 commutes. We define $h_\lambda := j \circ g_\lambda$ for all $\lambda \in \Lambda$, consequently the triangle (1) commutes. Moreover, h_λ is the unique such morphism; if we assume we have $m : Y_\lambda \to B$ such that the triangle (1) commutes, we have $m \circ f_\delta = h_\lambda \circ f_\delta$ for all $\delta \in \Delta_\lambda$ and so $m = h_\lambda$. Thus $\{X_\delta \xrightarrow{f_\delta} Y_\lambda\}_{\delta \in \Delta_\lambda}$ is a bounded coproduct for each $\lambda \in \Lambda$. Note that in both cases we have used the fact that f_δ's are jointly epic. This concludes the proof of (4).

To prove that (5) holds, suppose that $\{X_\delta \xrightarrow{f_\delta} Y | \lambda \in \Lambda \text{ and } \delta \in \Delta_\lambda\}$ is a bounded coproduct. Now define the objects Z_λ as the direct sums $\bigoplus_{\delta \in \Delta_\lambda} X_\delta$ for each $\lambda \in \Lambda$. Further, define the morphisms $g_\delta : X_\delta \to Z_\lambda$ to be the coprojections, and the morphisms h_λ the coprojections from each Z_λ to Y. It now follows that $h_\lambda \circ g_\delta$ are the coprojections from each X_δ to Y, hence $h_\lambda \circ g_\delta = f_\delta$, as required.

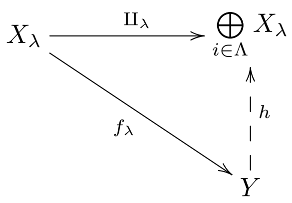
If $\bigoplus_{\lambda \in \Lambda} X_\lambda$ is a direct sum, and Z_δ is the trivial vector space for all $\delta \in \Delta$, then clearly $\bigoplus_{\lambda \in \Lambda} X_\lambda \oplus \bigoplus_{\delta \in \Delta} Z_\delta \simeq \bigoplus_{\lambda \in \Lambda} X_\lambda$. This implies that (6) holds. Conversely, if $\bigoplus_{\lambda \in \Lambda} X_\lambda$ and $\bigoplus_{\delta \in \Delta} Z_\delta$ are direct sums, and we have $\bigoplus_{\lambda \in \Lambda} X_\lambda \oplus \bigoplus_{\delta \in \Delta} Z_\delta \simeq \bigoplus_{\lambda \in \Lambda} X_\lambda$, then Z_δ must be the trivial vector space for all $\delta \in \Delta$; and so (7) holds.

For (8), suppose $f : X \xrightarrow{\sim} Y$ is an isomorphism. Given any morphism $g : X \to B$, let $\bar{g} = g \circ f^{-1}$. Now the diagram



commutes. Moreover, since f has an inverse, $\bar{g}$ is the unique such morphism, making the diagram into a bounded coproduct.

For (9), let $\{X_\lambda \xrightarrow{f_\lambda} Y\}_{\lambda \in \Lambda}$ be in $\mathfrak{ds}$. Then by Lemma 2.2 there is an isomorphism h from Y to the direct sum of X_λ's such that the diagram

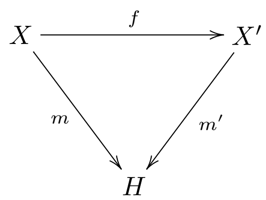


commutes for each $\lambda \in \Lambda$. Now if $f_{\lambda_1} = f_{\lambda_2}$ for some distinct λ_1 and λ_2, then $\amalg_{\lambda_1} = \amalg_{\lambda_2}$, which in turn implies $\amalg_{\lambda_1}(x) = \amalg_{\lambda_2}(x) = 0$ for all $x \in X_{\lambda_1} = X_{\lambda_2}$ by Definition 4.3. It then follows that $X_{\lambda_1} = X_{\lambda_2}$ is trivial.

6 Manuals in Hilb

Next we state what a manual is in **Hilb** and prove that the axioms of Section 3 are satisfied.

Given a Hilbert space H, we define a category $\mathfrak{M}_H$ as follows. Let (X, m) denote a pair of a Hilbert space X and a choice of an isometry $m : X \to H$ in **Hilb**, provided such a map exists. Note that a linear isometry is necessarily monic. The objects of $\mathfrak{M}_H$ are all such pairs (X, m). If it is clear from the context which map is in question, we will sometimes omit it and write X for an object in $\mathfrak{M}_H$. Given two objects (X, m) and (X', m') of $\mathfrak{M}_H$, define a morphism from (X, m) to (X', m') as the map $f : X \to X'$ in **Hilb** such that the diagram



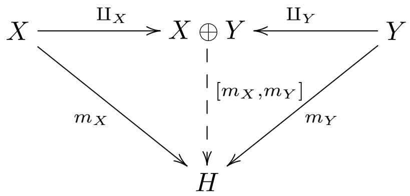
of **Hilb** commutes. Note that such f, if it exists, is necessarily unique, justifying the expression 'the map' in the definition above. Moreover, such f is itself an isometry, since for $x, y \in X$ we have

$$\big(f(x), f(y)\big)_{X'} = \big((m' \circ f)(x), (m' \circ f)(y)\big)_H = \big(m(x), m(y)\big)_H = \big(x, y\big)_X.$$

Define the functor $\mathfrak{F} : \mathfrak{M}_H \to$ **Hilb** by $\mathfrak{F}(X, m) = X$ and by mapping the morphisms in $\mathfrak{M}_H$ to themselves in **Hilb**. We now claim that $(\mathfrak{M}_H, \mathfrak{F})$ is a manual in **Hilb**, moreover, it is rich and completely coherent. Before proving this, we will need some preliminary results.

Proposition 6.1. *The objects (X, m_X) and (Y, m_Y) in $\mathfrak{M}_H$ are $\mathfrak{M}_H$-orthogonal if and only if the images of m_X and m_Y are orthogonal as vector subspaces of H.*

Proof. First suppose that $\mathrm{Im}(m_X) \perp \mathrm{Im}(m_Y)$. Then the unique morphism $[m_X, m_Y]$ in the diagram



is monic and an isometry by the definition of direct sum. Note that we have introduced the notation $[m_X, m_Y]$ for the unique map from $X \oplus Y$ to H defined by $(x, y) \mapsto m_X(x) + m_Y(y)$. Consequently, the diagram $X \xrightarrow{\amalg_X} X \oplus Y \xleftarrow{\amalg_Y} Y$ is in $\mathfrak{M}_H$, and so $X \perp_{\mathfrak{M}_H} Y$.

Now suppose that $X \perp_{\mathfrak{M}_H} Y$, that is, the unique morphism $[m_X, m_Y]$ in the above diagram is an isometry. Now let $x \in X$ and $y \in Y$, and consider the inner product

$$\begin{aligned}
\big(m_X(x), m_Y(y)\big)_H &= \big(m_X(x) + m_Y(0), m_X(0) + m_Y(y)\big)_H \\
&= \big([m_X, m_Y](x, 0), [m_X, m_Y](0, y)\big)_H \\
&= \big((x, 0), (0, y)\big)_{X \oplus Y} \quad \text{(since } [m_X, m_Y] \text{ is an isometry)} \\
&= \big(x, 0\big)_X + \big(0, y\big)_Y \\
&= 0.
\end{aligned}$$

Since this holds for any $x \in X$ and $y \in Y$, the images are indeed orthogonal. □

Proposition 6.2. *An object (X, m_X) in $\mathfrak{M}_H$ is $\mathfrak{M}_H$-maximal if and only if m_X is an (isometric) isomorphism of Hilbert spaces.*

Proof. One direction is straightforward; if m_X is an isomorphism, then the only Hilbert space that can be added to X such that there still is an injection from the direct sum to H is the zero space. For the converse, assume that (X, m_X) is $\mathfrak{M}_H$-maximal, that is, for any object (Z, m_Z) in $\mathfrak{M}_H$, if $Z \perp_{\mathfrak{M}_H} X$ then Z is the zero space. By Proposition 6.1, this is equivalent to assuming that Z is the zero space whenever we have $\mathrm{Im}(m_X) \perp \mathrm{Im}(m_Z)$. To see that m_X is surjective, let Z be the subspace of H defined by $Z = \mathrm{Im}(m_X)^\perp$, and m_Z the inclusion of Z into H. We now have $\mathrm{Im}(m_X) \perp \mathrm{Im}(m_Z)$ by definition, and so $Z = \mathrm{Im}(m_X)^\perp$ is the zero space. Consequently $\mathrm{Im}(m_X) = H$, and so m_X is an isomorphism, as required. □

Maximal objects in $\mathfrak{M}_H$ are thus precisely the Hilbert spaces isomorphic to H, in particular, H is maximal.

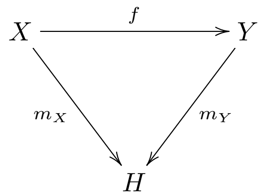
Proposition 6.3. *The objects (X, m_X) and (Y, m_Y) in $\mathfrak{M}_H$ are $\mathfrak{M}_H$-equivalent if and only if m_X and m_Y have the same image.*

Proof. This is a straightforward consequence of Proposition 6.1. Let (Z, m_Z) be an object in $\mathfrak{M}_H$. If $\mathrm{Im}(m_X) = \mathrm{Im}(m_Y)$, then manifestly $\mathrm{Im}(m_Z) \perp \mathrm{Im}(m_X)$ if and only if $\mathrm{Im}(m_Z) \perp \mathrm{Im}(m_Y)$. Conversely, if $X \simeq_{\mathfrak{M}_H} Y$, then $\mathrm{Im}(m_Z) \perp \mathrm{Im}(m_X)$ if and only if $\mathrm{Im}(m_Z) \perp \mathrm{Im}(m_Y)$, since this is the case for any object Z in $\mathfrak{M}_H$, we must have $\mathrm{Im}(m_X) = \mathrm{Im}(m_Y)$. □

Note that the Proposition 6.3 entails that $\mathfrak{M}_H$-equivalent objects are isomorphic as vector spaces.

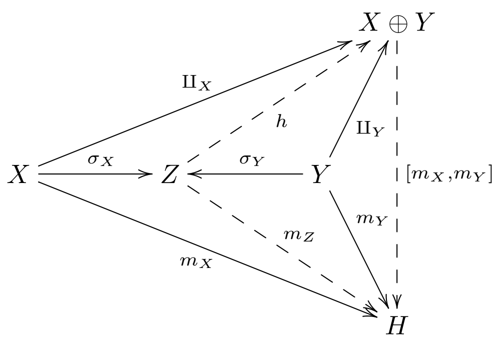
We are now ready to prove our claim that $(\mathfrak{M}_H, \mathfrak{F})$ is a **Hilb**-manual. First, we note that (10) holds by uniqueness of the morphism between objects of $\mathfrak{M}_H$. Since there is always a map from the zero space to any Hilbert space (and it is trivially an isometry), it is an object of $\mathfrak{M}_H$ and so (11) holds. The zero space is also the initial object in $\mathfrak{M}_H$, implying (12) is satisfied.

In order to show that (13) holds, assume there is a morphism $f : (X, m_X) \to (Y, m_Y)$ in $\mathfrak{M}_H$, hence the diagram



is commutative. We need to show that $Z \perp_{\mathfrak{M}_H} Y$ implies $Z \perp_{\mathfrak{M}_H} X$ for every object (Z, m_Z) in $\mathfrak{M}_H$, that is, by Proposition 6.1, $\mathrm{Im}(m_X) \perp \mathrm{Im}(m_Z)$ whenever $\mathrm{Im}(m_Y) \perp \mathrm{Im}(m_Z)$. It is thus sufficient to show that $\mathrm{Im}(m_X) \subseteq \mathrm{Im}(m_Y)$. Suppose that $h \in H$ is an element of $\mathrm{Im}(m_X)$, hence there is an $x \in X$ such that $m_X(x) = h$. This implies that $m_Y(f(x)) = h$, and so h is an element of $\mathrm{Im}(m_Y)$, as required.

For (14), suppose that the objects (X, m_X) and (Y, m_Y) of $\mathfrak{M}_H$ are $\mathfrak{M}_H$-orthogonal, that is, the object $X \oplus Y$ in the orthogonal sum diagram $X \xrightarrow{\amalg_X} X \oplus Y \xleftarrow{\amalg_Y} Y$ of **Hilb** is also an object of $\mathfrak{M}_H$. We will show that $X \oplus Y = X \oplus_{\mathfrak{M}_H} Y$. Suppose that the diagram $(X, m_X) \xrightarrow{\sigma_X} (Z, m_Z) \xleftarrow{\sigma_Y} (Y, m_Y)$ is in $\mathfrak{M}_H$ and $X \xrightarrow{\sigma_X} Z \xleftarrow{\sigma_Y} Y$ is an orthogonal sum diagram of **Hilb**. We need to show that the unique morphism $h : Z \to X \oplus Y$ of axiom (3) is in $\mathfrak{M}_H$. This follows neatly by drawing a diagram.



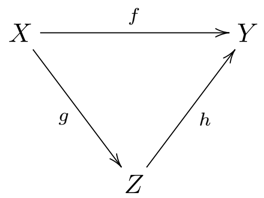
Now every subdiagram is either part of a coproduct or a diagram defining a morphism between objects of $\mathfrak{M}_H$, consequently, each of them commutes and so the entire diagram commutes. In particular, we have $m_Z = [m_X, m_Y] \circ h$, implying that h is in $\mathfrak{M}_H$.

Next, assume that $Z = X \oplus_{\mathfrak{M}_H} Y$ for objects (X, m_X) and (Y, m_Y) in $\mathfrak{M}_H$. To prove that (15) is satisfied, we need to show that $X \perp_{\mathfrak{M}_H} W$ and $Y \perp_{\mathfrak{M}_H} W$ imply $Z \perp_{\mathfrak{M}_H} W$ for any object (W, m_W) in $\mathfrak{M}_H$. By proposition 6.1, we have to show that $\mathrm{Im}(m_X) \perp \mathrm{Im}(m_W)$ and $\mathrm{Im}(m_Y) \perp \mathrm{Im}(m_W)$ imply $\mathrm{Im}\big([m_X, m_Y]\big) \perp \mathrm{Im}(m_W)$. But $\mathrm{Im}\big([m_X, m_Y]\big)$ is just the direct sum of $\mathrm{Im}(m_X)$

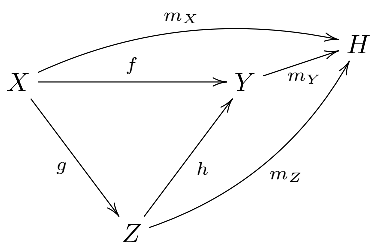
and $\mathrm{Im}(m_Y)$, which is by assumption also a subspace of H. If $\mathrm{Im}(m_W)$ is orthogonal to two subspaces of H, it is also orthogonal to their direct sum.

For (16), first assume that the objects (X, m_X) and (Y, m_Y) are $\mathfrak{M}_H$-equivalent, which by Proposition 6.3 means that $\mathrm{Im}(m_X) = \mathrm{Im}(m_Y)$. Let $Z = \mathrm{Im}(m_X)^\perp = \mathrm{Im}(m_Y)^\perp$ and m_Z the inclusion of Z into H. (Z, m_Z) is then an object of $\mathfrak{M}_H$, and we have $\mathrm{Im}(m_X) \perp \mathrm{Im}(m_Z)$, which by Proposition 6.1 implies that $X \perp_{\mathfrak{M}_H} Z$ and $Y \perp_{\mathfrak{M}_H} Z$. Moreover, by Proposition 6.2 $X \oplus_{\mathfrak{M}_H} Z$ and $Y \oplus_{\mathfrak{M}_H} Z$ are $\mathfrak{M}_H$-maximal, since $X \oplus_{\mathfrak{M}_H} Z \simeq H$. Conversely, suppose there is an object (Z, m_Z) in $\mathfrak{M}_H$ such that $\mathrm{Im}(m_X) \perp \mathrm{Im}(m_Z)$, $\mathrm{Im}(m_Y) \perp \mathrm{Im}(m_Z)$ and $X \oplus_{\mathfrak{M}_H} Z \simeq Y \oplus_{\mathfrak{M}_H} Z \simeq H$. This implies that $H = \mathrm{Im}(m_X) \oplus \mathrm{Im}(m_Z) = \mathrm{Im}(m_Y) \oplus \mathrm{Im}(m_Z)$, and so $\mathrm{Im}(m_X) = \mathrm{Im}(m_Y)$ by uniqueness of orthogonal complements.

For (17), suppose that the diagram

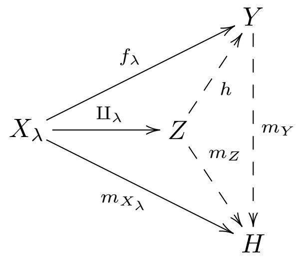


of **Hilb** commutes. Further, suppose that there are maps m_X, m_Y and m_Z such that (X, m_X), (Y, m_Y) and (Z, m_Z) are objects in $\mathfrak{M}_H$, and $f : (X, m_X) \to (Y, m_Y)$ and $h : (Z, m_Z) \to (Y, m_Y)$ morphisms between these objects. We want to show that $g : (X, m_X) \to (Z, m_Z)$ is also a morphism in $\mathfrak{M}_H$. This amounts to showing that the outer triangle in the diagram



commutes. This immediately follows from commutativity of the three smaller triangles.

We will next show that $(\mathfrak{M}_H, \mathfrak{F})$ is completely coherent, that is, (19) holds. Given an infinite family $(X_\lambda, m_{X_\lambda})_{\lambda \in \Lambda}$ of pairwise $\mathfrak{M}_H$-orthogonal objects in $\mathfrak{M}_H$, let $Z = \bigoplus_{\lambda \in \Lambda} X_\lambda$ be the orthogonal sum of X_λ's in **Hilb**. We will show that $Z = \bigoplus_{\lambda \in \Lambda}^{\mathfrak{M}_H} X_\lambda$. Since the images of m_{X_λ} are pairwise orthogonal, the unique morphism $m_Z : Z \to H$ is monic and an isometry by its definition. It is thus sufficient to show that for any orthogonal sum diagram $\{X_\lambda \xrightarrow{f_\lambda} Y\}_{\lambda \in \Lambda}$ of **Hilb** which can be pulled back to $\mathfrak{M}_H$, the unique morphism h of the diagram



is in $\mathfrak{M}_H$. By commutativity of the outer triangle and the two inner triangles on the left, we have $m_Y \circ h \circ \amalg_\lambda = m_Z \circ \amalg_\lambda$ for all $\lambda \in \Lambda$. Since the $\amalg_\lambda$'s are jointly epic, we have that $m_Y \circ h = m_Z$, as required.

In order to prove that $(\mathfrak{M}_H, \mathfrak{F})$ is rich, that is, (18) holds, we will need two intermediate results given by Lemmas 6.4 and 6.5.

Lemma 6.4. *The image of a bounded, linear isometry of Hilbert spaces is complete.*

Proof. Let $f : Y \to X$ be a bounded, isometric linear map between Hilbert spaces Y and X. Given a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in the image of f, we have to show that there exists an $x \in \operatorname{Im}(f)$ such that $x_n \to x$. Since the sequence is in the image of f, thre is a sequence $\{y_n\}_{n\in\mathbb{N}}$ in Y such that $f(y_n) = x_n$ for all n. Since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy, for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$\begin{aligned} n, m \geq N &\implies \|x_n - x_m\| < \epsilon \\ &\implies \|f(y_n - y_m)\| < \epsilon \\ &\implies \|y_n - y_m\| < \epsilon \quad \text{(since } f \text{ is an isometry)}, \end{aligned}$$

hence showing that $\{y_n\}_{n\in\mathbb{N}}$ is Cauchy. Being a Hilbert space Y is complete, and so there exists $y \in Y$ such that $y_n \to y$. Because f is a bounded linear map, it is continuous, hence we must have $f(y_n) \to f(y)$ in X, that is, $x_n \to x$, where $x = f(y) \in \operatorname{Im}(f)$. □

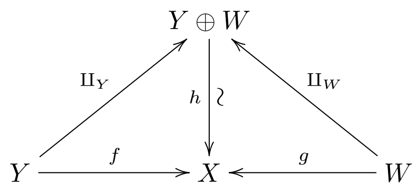
For a proof of the following, see Halmos [3].

Lemma 6.5. *[3, ch. 16, p. 74, Problem 134.] If $f : Y \to X$ is a bounded linear map between Hilbert spaces Y and X, then there exists a partial isometry $u : Y \to X$ and a positive endomorphism p of Y such that $f = u \circ p$. The maps u and p can be found such that* $\ker(u) = \ker(p)$*, and this additional condition uniquely determines them.*

Such unique representation of f as composition of u and p is called the *polar decomposition* of f.

Remark 6.6. We observe that if f is monic, then both u and p are monic. To see this, suppose we have some maps $g_1, g_2 : Z \rightrightarrows Y$ such that $p \circ g_1 = p \circ g_2$. This implies $u \circ p \circ g_1 = u \circ p \circ g_2$, but $u \circ p = f$, which we are assuming to be monic, hence $g_1 = g_2$, showing that p is monic. Consequently, $\ker(p) = \ker(u) = 0$, and so u is also monic. Note that since u is a partial isometry, it is an isometry on the orthogonal complement of its kernel, and so u being monic is a sufficient condition for it to be an isometry.

In order to show that (18) is satisfied, suppose that (X, m_X) is an object in $\mathfrak{M}_H$ and $f : Y \to X$ is an embedding in **Hilb**. We need to find an $\mathfrak{M}_H$-embedding $f' : (Y', m_{Y'}) \to (X, m_X)$ such that f and f' are equivalent in **Hilb**. We first note that the image of f is complete, this follows from f being an embedding. In detail, there is a Hilbert space W and a morphism g such that $Y \xrightarrow{f} X \xleftarrow{g} W$ is an orthogonal sum, hence there is an isomorphism h such that the diagram



commutes. In particular, we have $f = h \circ \amalg_Y$. Since h is an isomorphism, it maps complete sets to complete sets; since $\amalg_Y$ is an isometry, completeness of $\mathrm{Im}(\amalg_Y)$ follows by Lemma 6.4, thus, $\mathrm{Im}(f) = \mathrm{Im}(h \circ \amalg_Y)$ is complete. Note that we have also shown that f is monic.

Let us next write f as its polar decomposition, $f = u \circ p$. Note that since f is monic, it follows by Remark 6.6 that u is an isometry and p is monic. Furthermore, we claim that the image of p is complete and therefore a Hilbert space. To show this, we argue by contradiction; suppose that $\mathrm{Im}(p)$ is not complete, that is, there is a sequence $\{y_n\}_{n\in\mathbb{N}}$ in Y such that $\{p(y_n)\}_{n\in\mathbb{N}}$ is Cauchy but not convergent. Since u is an isometry, we therefore have that the sequence $\{u \circ p(y_n)\}_{n\in\mathbb{N}} = \{f(y_n)\}_{n\in\mathbb{N}}$ is Cauchy but does not converge in $\mathrm{Im}(u \circ p) = \mathrm{Im}(f)$, a contradiction.

If we now take $Y' = \mathrm{Im}(p)$ and $m_{Y'} = m_X \circ u|_{Y'}$, then $f' = u|_{Y'}$ is the desired $\mathfrak{M}_H$-embedding, where we have denoted by $u|_{Y'}$ the restriction of u to the image of p. It is easy to see that $u|_{Y'}$ is indeed an $\mathfrak{M}_H$-embedding by letting $Z = \mathrm{Im}(u|_{Y'})^{\perp}$ and the map from Z to X the inclusion of Z into X. To see that f and $u|_{Y'}$ are equivalent, it is sufficient to note that p is an isomorphism by the choice of codomain, and that we have $f = u \circ p = u|_{Y'} \circ p$. This concludes the proof of (18).

We have thus shown that $(\mathfrak{M}_H, \mathfrak{F})$ is rich, completely coherent manual of **Hilb**. We conclude this section with two observations.

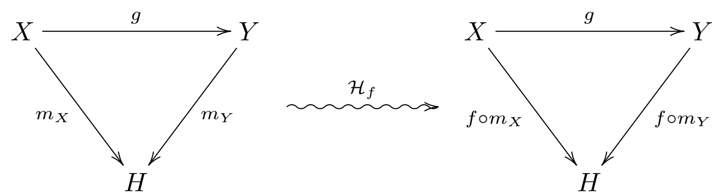
Firstly, it is crucial in the definition of $\mathfrak{M}_H$ that we require the morphism m_X from X to H to be an isometry. This is what guarantees that $\mathfrak{M}_H$-orthogonality corresponds to the orthogonality of images. We could try to loosen this condition and require the morphism m_X to be merely monic, making X a subobject of H, which would result in $\mathfrak{M}_H$-orthogonality amounting to linear independence of images. It is then straightforward to see that axiom (15) no longer holds.

Although the condition on m_X cannot be weakened, it is not the strictest one we can choose. A special case of a manual in **Hilb** is obtained by requiring the morphism m_X to be the actual inclusion of X into H. In this case all the maps between objects of the manual are also inclusions, and the manual consists of all the subspaces of H, and there is a map between two subspaces if and only if one contains the other.

7 Unitary Operators on Hilbert Spaces

As suggested by the concluding remarks of the previous section, $(\mathfrak{M}_H, \mathfrak{F})$ captures the subspace structure of H. In particular, it contains all the choices of orthogonal bases for H. What we are interested in are the interactions between the orthogonal bases; more precisely, given an operator on H, we are asking what kind of transformation it induces on $\mathfrak{M}_H$. We would like the operator to preserve orthogonality, which suggests we should consider a unitary map, or at least an isometry.

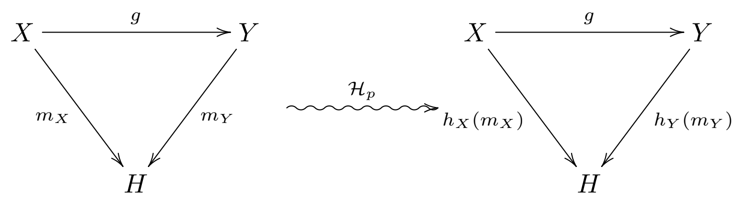
Let f be a unitary endomorphism of H. Now f induces an endofunctor $\mathcal{H}_f$ of $\mathfrak{M}_H$ defined as follows. For every object (X, m_X) of $\mathfrak{M}_H$, let $\mathcal{H}_f(X, m_X) = (X, f \circ m_X)$ and let $\mathcal{H}_f$ be identity on morphisms. That is, the commutative diagram on the left is mapped to the commutative diagram on the right.



Note that the requirement that f is unitary (or at least an isometry) is necessary so that $f \circ m_X$ and $f \circ m_Y$ are again isometries. Thus the action of f is to 'reshuffle' the isometries between X and H, we will next make this more precise.

Let X be a Hilbert space for which there exists an isometry from X to H. Let us denote the set of all isometries from X to H by $\mathrm{Iso}(X, H)$. The unitary endomorphism f then induces a function $h_f : \mathrm{Iso}(X, H) \to \mathrm{Iso}(X, H)$ defined by $m \mapsto f \circ m$. We note that such h_f is a bijection. Injectivity follows by injectivity of f. To see surjectivity, suppose that $m : X \to H$ is an isometry. Since f is unitary, it has an inverse which is also unitary; if we thus define $n := f^{-1} \circ m$, we have $n \in \mathrm{Iso}(X, H)$ and $h_f(n) = m$. We have therefore shown that for each Hilbert space X, f induces a *permutation* of the set $\mathrm{Iso}(X, H)$. Note that for surjectivity of h_f it is indeed necessary for f to be unitary rather than merely an isometry.

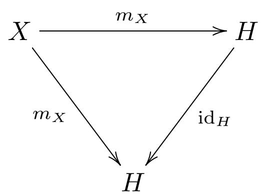
We immediately ask if the converse is true. That is, given a choice of a permutation of $\mathrm{Iso}(X, H)$ for every Hilbert space X, is the induced endomorphism of H unitary? The answer turns out to be yes if we choose the permutations consistently. Let h_X be a choice of a permutation of $\mathrm{Iso}(X, H)$. Let $\mathcal{H}_p$ be a choice of such permutation for every Hilbert space. Moreover, if X and Y are Hilbert spaces, we require that the diagram on the right commutes whenever the one on the left commutes.



This again, ensures that $\mathcal{H}_p$ is an endofunctor of $\mathfrak{M}_H$ defined by

$$\mathcal{H}_p\big((X, m_X) \xrightarrow{g} (Y, m_Y)\big) = (X, h_X(m_X)) \xrightarrow{g} (Y, h_Y(m_Y)).$$

The condition to preserve commutativity turns out to be so strict that it is in fact sufficient to specify $h_H(\mathrm{id}_H)$, which fixes all the other permutations. Indeed, if X is any Hilbert space from which there exists an isometry to H, then the diagram



commutes for any isometry $m_X : X \to H$. We must therefore have $h_X(m_X) = h_H(\mathrm{id}_H) \circ m_X$ for all $m_X \in \mathrm{Iso}(X, H)$. In particular, we have $h_H(m_H) = h_H(\mathrm{id}_H) \circ m_H$ for all $m_H \in \mathrm{Iso}(H, H)$. We now define $f \coloneqq h_H(\mathrm{id}_H)$ and claim that it is unitary. Since $h_H(\mathrm{id}_H)$ is an isometry, it is sufficient to show that f is surjective. Thus suppose we have $x \in H$, then there exists some isometry whose image contains x. Since h_H is a bijection, there exists an $m \in \mathrm{Iso}(H, H)$ such that $\mathrm{Im}(h_H(m))$ contains x. But we know that $h_H(m) = h_H(\mathrm{id}_H) \circ m$, and so x must also be contained in the image of h_H.

Observe that these two processes are mutually inverse; given a unitary f, the permutation h_H is defined by $m \mapsto f \circ m$, and so $h_H(\mathrm{id}_H) = f$; and conversely, the permutation of $\mathrm{Iso}(X, H)$ induced by $h_H(\mathrm{id}_H)$ sends each m_X to $h_H(\mathrm{id}_H) \circ m_X$, which is the original permutation we started with. Thus every unitary operator on H can be thought of as a choice of an isometry to which the identity morphism of H is mapped, which then determines a rearrangement of isometries on H.

Since $\mathfrak{M}_H$-orthogonality corresponds to orthogonality of images in H, the functors $\mathcal{H}_f$ and $\mathcal{H}_p$ necessarily map $\mathfrak{M}_H$-orthogonal sums to $\mathfrak{M}_H$-orthogonal sums, hence preserving the structure of the manual.

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