Some notes on path categories and globular weak ω-groupoids

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Abstract

We translate the construction in Bourke [4] from identity type categories to path categories to obtain an 'iterated path object' for any object in a path category. We indicate why Bourke's proof that every object in an identity type category carries a Grothendieck weak ω-groupoid structure does not straightforwardly carry over to path categories. We indicate possible constructions that could make the proof work.

1 Introduction

Given a type A in a Martin-Löf type theory, for any pair of elements $a : A$ and $b : A$ there is an associated identity type $\mathrm{Id}_A(a, b)$. Intuitively, every element in $\mathrm{Id}_A(a, b)$ corresponds to a *proof* that a and b are equal (if no such proof exists, then the identity type is not populated by any element). Since the work by Hofmann and Streicher [5], viewing types as spaces and the identity types as spaces of paths (or higher homotopies) has provided a fruitful perspective. Hofmann and Streicher show that the identity types of A determine an equivalence relation on the elements of A, namely, one declares that $a \sim b$ if and only if $\mathrm{Id}_A(a, b)$ is populated. Furthermore, they show that A has the structure of a groupoid, provided that we take the equality up to higher identity types. Another highly influential insight of Hofmann and Streicher is to construct a model of type theory where types are in fact interpreted as groupoids. The identity type $\mathrm{Id}_A(a, b)$ is then interpreted as the (discrete) groupoid consisting of all morphisms between

a and b in the groupoid A [1, p. 2]. Since there can multiple non-identical morphisms between a and b, this model shows that it is impossible to show that all elements of $\mathrm{Id}_A(a,b)$ are equal (even if we take equality up to a higher identity type).

This suggests a startlingly spatial interpretation of types: types may be viewed as spaces, while their elements as points in the space. The identity types then become path spaces, higher identity types homotopies between paths, homotopies between homotopies and so on. But a space is more than merely a groupoid; not only every path between two points has an inverse path (up to a homotopy), but any homotopy between paths has an inverse homotopy (up to a homotopy between homotopies). There is no reason to stop at any point, and indeed: any topological space gives rise to an infinity groupoid in this way. Inspired by this spatial analogy, a question arises: can we mimic this infinite tower of paths in an arbitrary model of type theory?

It would indeed be plausible to assume that the answer to this question is yes. For suppose we have two elements in the identity type $p, q : \mathrm{Id}_A(a,b)$. Since p and q are themselves types, we may iterate the application of the construction rule of the identity type to obtain the identity type $\mathrm{Id}_{\mathrm{Id}_A(a,b)}(p,q)$ (if we think of p and q as paths between a and b, then the elements in this iterated identity type are paths (homotopies) between paths).

It was indeed shown by van den Berg and Garner [2] that every object in the syntactic category of intensional type theory admits the structure of a weak ω-groupoid. They use the definition of a weak ω-category (and hence that of a weak ω-groupoid) due to Batanin (more precisely, the version from Leinster [6]). Bourke [4] reformulates their proof using the definition of a weak ω-groupoid that first appeared in Grothendieck's manuscript *Pursuing Stacks*, and was simplified by Maltsiniotis [7]. Following Maltsiniotis and Bourke, we term this construction a *Grothendieck ω-groupoid*. Like van den Berg and Garner, Bourke uses the so-called *identity type categories* as models of type theory. These categories in particular form a weak factorisation system.

The notion of a *path category* was first introduced by van den Berg and Moerdijk [3] as a strengthening of Brown's *category of fibrant objects*. As shown by van den Berg [1], path categories are models for Martin-Löf type theories with a weakened version of the computation rule for the identity type, where definitional equality is replaced by propositional equality. Since all equalities in a weak ω-groupoid hold up to a higher homotopy, it is reasonable to expect that objects in path categories will carry a Grothendieck weak ω-groupoid structure akin to the one in identity type categories. The aim here is to pave the way for showing this.

Our strategy is to adapt the proof by Bourke [4] from the context of identity type categories to path categories. The proof does not straightforwardly carry over to our context, since the lifting property of a weak factorisation system, which is exploited by Bourke's proof, does not hold in path categories. A weaker property, however, does hold. Namely, as shown by van den Berg and Moerdijk [3], a commutative square whose left side is a weak equivalence and right side a fibration does indeed have a diagonal filler, but only in the sense that the upper triangle commutes up to a fiberwise homotopy, while the lower one still commutes strictly (see Theorem 7).

2 Path categories

Definition 1. [Path category] Let $\mathcal{C}$ be a category, and suppose we have chosen two subclasses of maps of $\mathcal{C}$, denoted by $\mathfrak{F}$ and $\mathfrak{W}$. Maps in $\mathfrak{F}$ are called *fibrations*; and the maps in $\mathfrak{W}$ are called *weak equivalences*. Further, the maps that are both in $\mathfrak{F}$ and $\mathfrak{W}$ are called *acyclic fibrations* and their class is denoted by $\mathfrak{A}$. We say that the triple $(\mathcal{C}, \mathfrak{F}, \mathfrak{W})$ is a *path category* if the following conditions are satisfied.

(1) Fibrations $\mathfrak{F}$ are closed under composition.

(2) The category $\mathcal{C}$ has all pullbacks of fibrations along arbitrary maps, and a pullback of a fibration is itself a fibration.

(3) The category $\mathcal{C}$ has a terminal object 1, and for any object X the unique map $X \to 1$ is a fibration.

(4) Weak equivalences satisfy *2-out-of-6*, that is, if

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

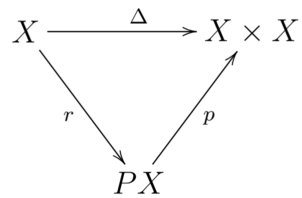
are maps such that $gf, hg \in \mathfrak{W}$, then also $f, g, h, hgf \in \mathfrak{W}$.

(5) Every isomorphism is an acyclic fibration.

(6) Every acyclic fibration has a section.

(7) A pullback of an acyclic fibration is itself an acyclic fibration.

(8) For every object $X \in \mathcal{C}$ there is (at least one) object $PX \in \mathcal{C}$ such that the diagonal $\Delta : X \to X \times X$ factors through PX as $\Delta = pr$, where $r \in \mathfrak{W}$ and $p \in \mathfrak{F}$. Such PX is called a *path object* of X.



Remarks.

1. We will usually refer to a path category just as $\mathcal{C}$, leaving fibrations and weak equivalences implicit.

2. We often denote the map p in condition 8 by its components, $p = (s, t)$.

3. Note that the class of isomorphisms in any category satisfies the condition 4.

4. Conditions 2 and 3 imply that $\mathcal{C}$ has all finite products, so that the statement of condition 8 is meaningful.

5. Acyclic fibration are sometimes called *trivial* fibrations.

Definition 2. [Homotopy, homotopic maps] Let $\mathcal{C}$ be a path category, and suppose $f, g : Y \to X$ are parallel maps. We say that f and g are *homotopic* if there is a path object PX with fibration $(s, t) : PX \to X \times X$ and a map $h : Y \to PX$ (called a *homotopy*) such that $sh = f$ and $th = g$. In this case we write $f \simeq g$, or $h : f \simeq g$ to emphasise the choice of homotopy.

It can be shown that the homotopy relation is independent of the choice of the path object and that it is in fact an equivalence relation on $Hom(Y, X)$ (see van den Berg and Moerdijk [3, pp. 3142-3145]). Moreover, the homotopy relation is a *congruence* in the following sense.

Theorem 3 (Theorem 2.14 in van den Berg and Moerdijk [3])**.** *If f and g are parallel maps, then $f \simeq g$ implies $fk \simeq gk$ and $lf \simeq lg$ for any maps k and l for which this composition makes sense.*

While condition 8 only requires that the diagonals factor as a weak equivalence followed by a fibration, the following may be shown for any path category.

Proposition 4 (Proposition 2.3 in van den Berg and Moerdijk [3]). *Any morphism $f : Y \to X$ in a path category factors as $f = p_f w_f$, where $p_f \in \mathfrak{F}$ and w_f is a section of an acyclic fibration (and hence $w_f \in \mathfrak{W}$).*

We may use this fact to define fiberwise path objects, which are in turn used to define fiberwise homotopies.

Definition 5 (Fiberwise path object). Let $p : B \to A$ be a fibration, and denote the pullback of p along itself by $B \times_A B$. Then the fiberwise diagonal $\Delta_A = (\mathrm{id}_B, \mathrm{id}_B)$ factors as a weak equivalence followed by a fibration

$$B \xrightarrow{r} P_A B \xrightarrow{(s,t)} B \times_A B$$

through some object $P_A B$. Any such $P_A B$ is called a *fiberwise path object* for B with respect to p.

Definition 6 (Fiberwise homotopy). Let $\mathcal{C}$ be a path category, and suppose $f, g : Y \to X$ are parallel maps. Further suppose that there is a fibration $p : X \to I$ such that $pf = pg$. We say that f and g are *fiberwise homotopic* with respect to p if there is a fiberwise path object $P_I X$ with fibration $(s, t) : P_I X \to X \times_I X$ and a map $h : Y \to P_I X$ (called a *fiberwise homotopy*) such that $sh = f$ and $th = g$. In this case we write $f \simeq_I g$, or $h : f \simeq_I g$.

Fibrations and weak equivalences admit the following lifting property.

Theorem 7 (Theorem 2.38 in van den Berg and Moerdijk [3]). *In a path category, suppose that we have a commutative square*

$$\begin{array}{ccc} D & \xrightarrow{g} & C \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{f} & A \end{array}$$

with $w \in \mathfrak{W}$ and $p \in \mathfrak{F}$. Then there is a map $l : B \to C$ such that $pl = f$ and $lw \simeq_A g$. Such a map is unique up to a fiberwise homotopy with respect to p.

3 Homotopy coslices

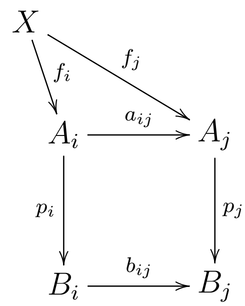
After constructing an iterated path object for an object X in an identity type category (as we shall do in Section 5), Bourke uses the coslice category with X as its apex to factor the iterated path object through it. This allows one to construct a globular theory which both has globular products of the required shape and is contractible (see sections 4 and 6). We could, in principle, do the same. However, the resulting globular theory would fail to be contractible, as a path category is not a weak factorisation system: the closest analogous lifting property we have is Theorem 7. Crucially, the upper triangle there does not commute strictly, and hence the resulting lifting is not a morphism in the strict coslise category as in Bourke's proof. This suggests a weakening to the category we use: the morphisms should commute up to a (fiberwise) homotopy. Here we define two candidate homotopy coslice categories for this.

Definition 8 (Homotopy coslice category)**.** Let $\mathcal{C}$ be a path category and let $A \in \mathcal{C}$. The *homotopy coslice category* $A \mathbin{/\!\!/} \mathcal{C}$ has as its objects equivalence classes of maps $A \to B$ in $\mathcal{C}$ under the homotopy relation, and a map from $[n] : A \to B$ to $[m] : A \to C$ is a morphism $k : B \to C$ such that $kn \simeq m$. Identities in $A \mathbin{/\!\!/} \mathcal{C}$ are simply the identities in $\mathcal{C}$, and composition is defined by pasting the triangles (that commute up to a homotopy).

Remark 9. Composition in a homotopy coslice category is well-defined due to Theorem 3. By a slight abuse of notation, we will almost exclusively refer to the objects of a homotopy coslice category by a representative from the equivalence class rather than by the entire class.

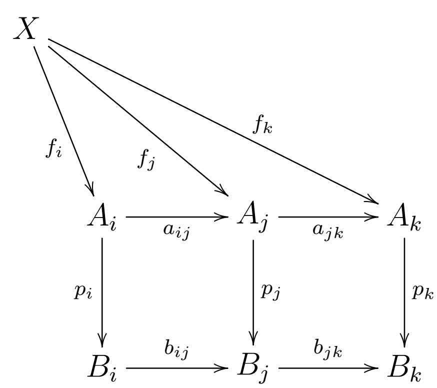
We will denote by $U : A \mathbin{/\!\!/} \mathcal{C} \to \mathcal{C}$ the forgetful functor from the homotopy coslice category to the base path category; it sends each object $[n] : A \to B$ to B and each morphism to itself (forgetting the commutativity condition).

Definition 10 (Fiberwise homotopy coslice category)**.** Let $\mathcal{C}$ be a path category and let $X \in \mathcal{C}$. An object in the *fiberwise homotopy coslice category* $X \mathbin{/\!\!/}^f \mathcal{C}$ consists of a fibration $p : A \to B$ in $\mathcal{C}$ and an equivalence class of maps $X \to A$ under the fiberwise homotopy relation with respect to p. A morphism from $(p_i : A_i \to B_i, f_i : X \to A_i)$ to $(p_j : A_j \to B_j, f_j : X \to A_j)$ is a pair of maps $(a_{ij} : A_i \to A_j, b_{ij} : B_i \to B_j)$ such that $p_j a_{ij} f_i = p_j f_j$, and in the diagram

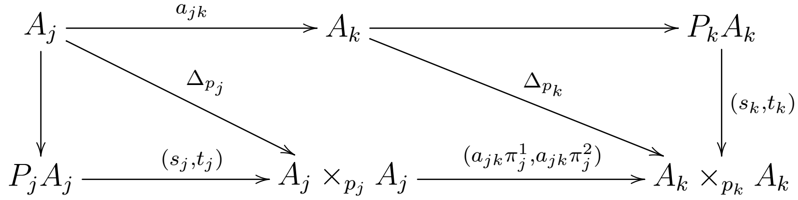


the triangle commutes up to a fiberwise homotopy with respect to p_j and the square commutes strictly. Composition is defined by pasting the triangles and squares.

Remark 11. Composition in a fiberwise homotopy coslice category is well-defined. For let $(a_{ij} : A_i \to A_j, b_{ij} : B_i \to B_j)$ and $(a_{jk} : A_j \to A_k, b_{jk} : B_j \to B_k)$ be morphisms. Then it is immediate that the rectangle in



commutes and that $p_k a_{jk} a_{ij} f_i = p_k f_k$. It remains to show $a_{jk} a_{ij} f_i \simeq_{p_k} f_k$. Let $P_j A_j$ and $P_k A_k$ be fiberwise path objects for A_j and A_k and let $h_j : X \to P_j A_j$ and $h_k : X \to P_k A_k$ the homotopies so that $s_j h_j = a_{ij} f_i$, $t_j h_j = f_j$, $s_k h_k = a_{jk} f_j$ and $t_k h_k = f_k$. We then have the following diagram



where the triangles factorise the diagonals with a weak equivalence followed by a fibration and hence commute, while the middle parallelogram commutes by construction. Since the leftmost map is a weak equivalence and the rightmost one a fibration, the diagram has a diagonal lifting $\alpha : P_j A_j \to P_k A_k$ making the lower triangle commute. It follows that

$$t_k \alpha h_j = a_{jk} t h_j = a_{jk} f_j$$

and

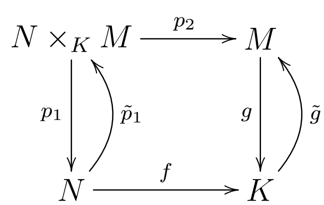
$$s_k \alpha h_j = a_{jk} a_{ij} f_i,$$

whence

$$a_{jk} a_{ij} f_i \simeq_{p_k} a_{jk} f_j \simeq_{p_k} f_k.$$

Proposition 12. *Let $n : A \to N$, $m : A \to M$ and $k : A \to K$ be objects in a homotopy coslice category $A /\!\!/ \mathcal{C}$, and suppose $f : n \to k$ and $g : m \to k$ are morphisms such that g is an acyclic fibration (in $\mathcal{C}$). Then the pullback of f and g exists in $A /\!\!/ \mathcal{C}$ and is mapped to the pullback of f and g in $\mathcal{C}$ by U.* (In short: $A /\!\!/ \mathcal{C}$ has pullbacks of acyclic fibrations, and these are created by U.)

Proof. We have the pullback



where p_1 is also an acyclic fibration by (7) of Definition 1. We have denoted the sections of g and p_1 by $\tilde{g}$ and $\tilde{p}_1$. Since both $fn \simeq k$ and $gm \simeq k$, we have $fn \simeq gm$. Now define $p := \tilde{p}_1 n$. This turns the maps p_1 and p_2 into maps in $A /\!\!/ \mathcal{C}$. Indeed, $p_1 p = n$, and

$$g p_2 p = f p_1 \tilde{p}_1 n = fn \simeq gm,$$

whence $p_2 p \simeq m$, as by Theorem 2.16 of van den Berg and Moerdijk [3] any section of a weak equivalence is a homotopy inverse. By the same theorem, and since homotopy inverses are unique up to a homotopy, this is the unique way to lift p_1 and p_2 to $A /\!\!/ \mathcal{C}$.

The only thing that remains to be checked is that the unique morphism into $N \times_K M$ induced by the universal property of the pullback in $\mathcal{C}$ is also a morphism in $A /\!\!/ \mathcal{C}$, which once more follows from the fact that p_1 has a homotopy inverse. □

4 Globular theories and weak ω-groupoids

This section is based on Section 2 of Bourke [4].

Definition 13. *The category of globes* $\mathbb{G}$ is the category freely generated by the graph

$$0 \underset{\tau_0}{\overset{\sigma_0}{\rightrightarrows}} 1 \underset{\tau_1}{\overset{\sigma_1}{\rightrightarrows}} \cdots \underset{\tau_{n-1}}{\overset{\sigma_{n-1}}{\rightrightarrows}} n \underset{\tau_n}{\overset{\sigma_n}{\rightrightarrows}} \cdots,$$

subject to the equations

$$\sigma_{n+1} \circ \sigma_n = \tau_{n+1} \circ \sigma_n$$
$$\tau_{n+1} \circ \tau_n = \sigma_{n+1} \circ \tau_n$$

for all $n \in \mathbb{N}$. Here 'freely generated' means we add an identity for every object and add as morphisms all the (formal) compositions, after which we quotient the class of morphisms by the identity and associativity relations as well as by the identities above.

Note that the equations in the definition of the category of globes imply that each homset $\mathbb{G}(n, m)$ contains exactly two elements whenever $n < m$. Namely, $\mathbb{G}(n, m) = \{\sigma_{n,m}, \tau_{n,m}\}$, where we abbreviate

$$\sigma_{n,m} \coloneqq \sigma_{m-1} \circ \cdots \circ \sigma_n$$

and similarly for $\tau_{n,m}$. If the domain and codomain of $\sigma_{n,m}$ and $\tau_{n,m}$ are clear from the context, we will simply write σ and τ.

If $\mathcal{C}$ is some category, we call a functor $A : \mathbb{G}^{op} \to \mathcal{C}$ an *ω-graph* or a *globular object* in $\mathcal{C}$. We write $s_n \coloneqq A(\sigma_n)$ and $t_n \coloneqq A(\tau_n)$, and similarly $s_{n,m} \coloneqq A(\sigma_{n,m})$ and $t_{n,m} \coloneqq A(\tau_{n,m})$, which we abbreviate by s and t if the context is clear.

A *table of dimensions* is a finite sequence of natural numbers $\bar{n} = (n_1, \dots, n_k)$ where k is odd and $n_{2i-1} > n_{2i} < n_{2i+1}$ for all $i = 1, \dots, \frac{k-1}{2}$. That is, the sequence has an odd number of elements, and all numbers at even positions are strictly less than their immediate neighbours. Each table of dimensions $\bar{n}$ determines a subcategory $\mathbb{G}^{\bar{n}}$ of $\mathbb{G}$ consisting of those objects which appear in $\bar{n}$, while $\mathbb{G}^{\bar{n}}(n_{2i}, n_{2i-1}) \coloneqq \{\tau\}$ and $\mathbb{G}^{\bar{n}}(n_{2i}, n_{2i+1}) \coloneqq \{\sigma\}$ for all i. The category $\mathbb{G}^{\bar{n}}$ contains no other non-identity morphisms. Hence $\mathbb{G}^{\bar{n}}$ looks like:

$$\begin{array}{ccccccccccc} & & n_2 & & & & n_4 & & \cdots & & n_{k-1} & & \\ & \overset{\tau}{\swarrow} & & \overset{\sigma}{\searrow} & & \overset{\tau}{\swarrow} & & \overset{\sigma}{\searrow} & & \overset{\tau}{\swarrow} & & \overset{\sigma}{\searrow} & \\ n_1 & & & & n_3 & & & & n_5 \quad n_{k-2} & & & & n_k \,. \end{array}$$

Definition 14. [Globular sum] Let $\mathcal{C}$ be a category and $D : \mathbb{G} \to \mathcal{C}$ a functor. Then each table of dimensions $\bar{n}$ determines a diagram $D^{\bar{n}} : \mathbb{G}^{\bar{n}} \to \mathcal{C}$ by restriction of D to $\mathbb{G}^{\bar{n}}$. The colimit of $D^{\bar{n}}$ (if exists) is called a *D-globular sum* (or just a globular sum) and is denoted by $D(\bar{n})$. If such a colimit exists for every table of dimensions $\bar{n}$, we say that $\mathcal{C}$ has all (D-)globular sums.

Dualising Definition 14, we obtain the definition of a *globular product*. Explicitly, for a globular object $A : \mathbb{G}^{op} \to \mathcal{C}$ and a table of dimensions $\bar{n}$, the limit of the diagram $A^{\bar{n}} : (\mathbb{G}^{op})^{\bar{n}} \to \mathcal{C}$, if it exists, is called an (A-)*globular product* and is denoted by $A(\bar{n})$.

Let $y : \mathbb{G} \to [\mathbb{G}^{op}, \mathbf{Set}]$ be the Yoneda embedding. Being a presheaf category, $[\mathbb{G}^{op}, \mathbf{Set}]$ is cocomplete and hence has all y-globular sums. Hence we may define the category Θ_0 as follows. The objects of Θ_0 are tables of dimensions, and

$$\Theta_0(\bar{n}, \bar{m}) := [\mathbb{G}^{op}, \mathbf{Set}](y(\bar{n}), y(\bar{m})).$$

Proposition 15. *The category Θ_0 is equivalent to the full subcategory of $[\mathbb{G}^{op}, \mathbf{Set}]$ whose objects are y-globular sums.*

Proof. Given $\bar{n} \in \Theta_0$, define $\iota(\bar{n}) := y(\bar{n})$ and let ι be identity on morphisms. This assignment is functorial, and F is full and faithful by definition. It is also essentially surjective on objects (in fact surjective), as every globular sum arises from a table of dimensions. □

Observe that the Yoneda embedding factors via Θ_0 as follows

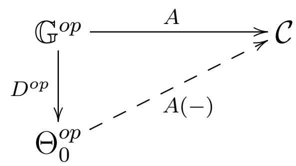
$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{\quad y \quad} & [\mathbb{G}^{op}, \mathbf{Set}] \\ {\scriptstyle D}\downarrow & \nearrow_{\iota} & \\ \Theta_0 & & \end{array},$$

where ι is as defined in the proof of Proposition 15, and $D : \mathbb{G} \to \Theta_0$ is given by $Dn := (n)$ on objects and by

$$D(n \xrightarrow{f} m) := y(n) \xrightarrow{f \circ -} y(m)$$

on morphisms. In fact the (dual of the) map D has a universal property as expressed in the following lemma. This is Lemma 2.1 in [4], where it is stated without a proof.

Lemma 16. *Let $A : \mathbb{G}^{op} \to \mathcal{C}$ be a globular object in $\mathcal{C}$ and suppose that $\mathcal{C}$ has all A-globular products. Then there is an essentially unique, globular product preserving extension $A(-) : \Theta_0^{op} \to \mathcal{C}$ such that the diagram*



commutes. On objects this extension is given by $\bar{n} \mapsto A(\bar{n})$.

Definition 17. A *globular theory* consists of

- a category $\mathbb{T}$ with $Ob(\mathbb{T}) = Ob(\Theta_0)$,
- a globular product preserving functor $J : \Theta_0^{op} \to \mathbb{T}$ which is identity on objects.

We shall write $Mod(\mathbb{T}, \mathcal{C})$ for the full subcategory of the functor category $[\mathbb{T}, \mathcal{C}]$ containing the globular product preserving functors, and call it the *category of* $\mathbb{T}$*-algebras in* $\mathcal{C}$. We then have a forgetful functor

$$U : Mod(\mathbb{T}, \mathcal{C}) \to [\mathbb{G}^{op}, \mathcal{C}]$$

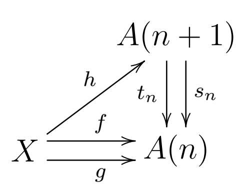
given by $X \mapsto X \circ J \circ D^{op}$ on objects. If $X \in Mod(\mathbb{T}, \mathcal{C})$ is such that $U(X) = A$ for some globular object A, we say that X is a $\mathbb{T}$*-algebra structure on* A.

Definition 18. Let $A : \mathbb{G}^{op} \to \mathcal{C}$ be a globular object and let $X \in \mathcal{C}$. Then

$$f, g : X \rightrightarrows A(n)$$

is a *parallel pair of* n*-cells in* A if either $n = 0$, or $s_{n-1}f = s_{n-1}g$ and $t_{n-1}f = t_{n-1}g$.

Definition 19. A *lifting* for a parallel pair of n-cells $f, g : X \rightrightarrows A(n)$ is a map $h : X \to A(n+1)$ such that the diagram



commutes serially, that is, we have $s_n h = f$ and $t_n h = g$.

Definition 20. A globular object $A : \mathbb{G}^{op} \to \mathcal{C}$ is *contractible* if every parallel pair of n-cells in A has a lifting, for all $n \in \mathbb{N}$.

We say that a globular theory $J : \Theta_0^{op} \to \mathbb{T}$ is contractible if its underlying globular object

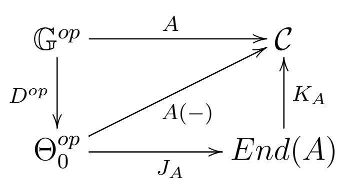
$$J \circ D^{op} : \mathbb{G}^{op} \to \mathbb{T}$$

is contractible.

Definition 21. A *Grothendieck weak ω-groupoid* is a $\mathbb{T}$-algebra for some contractible globular theory $J : \Theta_0^{op} \to \mathbb{T}$.

4.1 Endomorphism theories

Let $A : \mathbb{G}^{op} \to \mathcal{C}$ be a globular object and suppose $\mathcal{C}$ has all A-globular products. Let $A(-) : \Theta_0^{op} \to \mathcal{C}$ be the extension as in Lemma 16. We then define the category $End(A)$ by letting $Ob(End(A)) := Ob(\Theta_0)$ and $End(A)(\bar{n}, \bar{m}) := \mathcal{C}(A(\bar{n}), A(\bar{m}))$. Now $A(-)$ factors via $End(A)$ as follows:



where J_A is identity on objects and maps $f : \bar{n} \to \bar{m}$ to Af, and K_A assigns $\bar{n} \to A(\bar{n})$ and is identity on morphisms. Since $A(-)$ preserves globular products, so do J_A and K_A. Thus $J_A : \Theta_0^{op} \to End(A)$ is a globular theory. Moreover, we have $K_A \circ J_A \circ D^{op} = A$, so that K_A is a $End(A)$-algebra structure on A.

The following observation will be used to construct a weak ω-groupoid in path categories.

Lemma 22. *Let $A : \mathbb{G}^{op} \to \mathcal{C}$ be a globular object and suppose $\mathcal{C}$ has all globular products. Then $J_A : \Theta_0^{op} \to End(A)$ is contractible if and only if each parallel pair of m-cells $f, g : A(\bar{n}) \rightrightarrows A(m)$ whose domain is a globular product has a lifting for all $m \in \mathbb{N}$.*

Proof. The theory J_A is contractible if and only if the globular object $J_A \circ D^{op} \to End(A)$ is contractible, that is, if and only if every parallel pair of m-cells in $J_A \circ D^{op}$ has a lifting for all $m \in \mathbb{N}$. Since K_A is identity on morphisms and surjective on globular products, this is equivalent to: every parallel pair of m-cells in A whose domain is a globular product has a lifting for all $m \in \mathbb{N}$, which is as claimed. $\square$

In particular, if J_A is contractible, then K_A is a Grothendieck weak ω-groupoid.

5 The iterated path object

From now on and until the end of the paper, let $\mathcal{C}$ be some path category and X a fixed object in $\mathcal{C}$. Here we will inductively define a particular globular object $X_* : \mathbb{G}^{op} \to \mathcal{C}$ by using the structure of a path category. In Section 6 it will be shown than there is a contractible globular theory whose underlying globular object is X_*. During the induction, it will be useful to define a collection of objects $B_{n+1}X_*$ in $\mathcal{C}$, known as the $(n+1)$-*boundary* of an (inductively defined) n-graph.

We first define $X_*(0) \coloneqq X$ and $B_1 X_* \coloneqq X_*(0) \times X_*(0)$. Next, we define $X_*(1) \coloneqq PX$ as a path object of X, that is, we have a factorisation of the diagonal:

$$X_*(0) \xrightarrow{r_0} X_*(1) \xrightarrow{(s_0,t_0)} B_1 X_*$$

with $r_0 \in \mathfrak{W}$ and $p_0 \coloneqq (s_0, t_0) \in \mathfrak{F}$. Thus we may define $X_*\sigma_0 \coloneqq s_0$ and $X_*\tau_0 \coloneqq t_0$.

Now suppose that X_* and $B_n X_*$ as well as fibrations $p_{n-1} : X_*(n) \to B_n X_*$ are all defined up to some $n \geq 1$. For X_* this means that it is defined up to n on objects and up to $n-1$ on morphisms. We define $B_{n+1}X_*$ as the pullback:

$$\begin{array}{ccc} B_{n+1}X_* & \xrightarrow{k_n} & X_*(n) \\ {\scriptstyle q_n}\downarrow & & \downarrow{\scriptstyle p_{n-1}} \\ X_*(n) & \xrightarrow[p_{n-1}]{} & B_n X_*. \end{array} \tag{23}$$

Since p_{n-1} is a fibration, so are q_n and k_n.

Let $X_*(n+1)$ be a fiberwise path object for $X_*(n)$ with respect to p_{n-1}, that is, the fiberwise diagonal $\Delta_{p_{n-1}} : X_*(n) \to B_{n+1}X_*$ factors as

$$X_*(n) \xrightarrow{r_n} X_*(n+1) \xrightarrow{(s_n,t_n)} B_{n+1}X_*$$

with $r_n \in \mathfrak{W}$ and $p_n := (s_n, t_n) \in \mathfrak{F}$. As for the base case, we define $X_*\sigma_n := s_n$ and $X_*\tau_n := t_n$. This completes the construction of X_*. Note that $s_n = q_n p_n$ and $t_n = k_n p_n$, so that both s_n and t_n are fibrations. On the other hand, $s_n r_n = t_n r_n = \mathrm{id}_{X_*(n)}$, and since r_n is a weak equivalence, so are s_n and t_n. Thus we have $s_n, t_n \in \mathfrak{A}$.

Lemma 24. *The assignment $X_* : \mathbb{G}^{op} \to \mathcal{C}$ as defined above is functorial and hence defines a globular object in $\mathcal{C}$.*

Proof. Defining the action of X_* on identities as is necessary, the only thing we have to check is that X_* preserves the equations in Definition 13. That is, we need to show that

$$\begin{aligned} s_n s_{n+1} &= s_n t_{n+1} \\ t_n t_{n+1} &= t_n s_{n+1}. \end{aligned}$$

For $n \geq 1$, we have

$$s_n s_{n+1} = q_n p_n q_{n+1} p_{n+1} = q_n p_n k_{n+1} p_{n+1} = s_n t_{n+1},$$

where in the middle equality we used commutativity of the square in (23). The case $n = 0$ is almost identical, we just use π_0 instead of q_0. The second equality follows similarly. □

Any X_* so constructed is called an iterated path object of X.

6 The weak ω-groupoid structure

We are now ready to state our conjecture.

Conjecture 25. *Let X be an object in a path category $\mathcal{C}$ and $X_* : \mathbb{G}^{op} \to \mathcal{C}$ an iterated path object. Then there is a contractible globular theory $\mathbb{T}$ and a Grothendieck weak ω-groupoid which is a $\mathbb{T}$-algebra structure on X_*.*

Before discussing the possible proof strategies, we introduce some notation. In Section 5, we defined weak equivalences $r_n : X_*(n) \to X_*(n+1)$, whose compositions we denote by $r_{n,m} : X_*(n) \to X_*(m)$ for $n < m$, and we set $r_{n,n} = \mathrm{id}_{X_*(n)}$, dropping the subscripts when the domain and codomain are clear. For the case $n = 0$, we shall write $r^0_m := r_{0,m} : X_*(0) \to X_*(m)$ for $m \geq 0$.

Observe that for $m > n$ we have

$$\begin{aligned} sr^0_m &= s_n \cdots s_m r_m \cdots r_0 \\ &= s_n \cdots s_{m-1} r_{m-1} \cdots r_0 \\ &\quad \cdots \\ &= r_n \cdots r_0 \\ &= r^0_n, \end{aligned}$$

and similarly $tr^0_m = r^0_n$. This exhibits the maps r^0_m as a cone on X_* with apex $X_*(0)$. We use this to factor X_* as follows:

$$\begin{array}{ccc} \mathbb{G}^{op} & \xrightarrow{R} & X_*(0) \mathbin{/\!\!/} \mathcal{C} \\ & \searrow^{X_*} & \downarrow U \\ & & \mathcal{C} \end{array}$$

where $X_*(0) \mathbin{/\!\!/} \mathcal{C}$ is the homotopy coslice category, U is the forgetful functor, while R is defined by $n \mapsto r^0_n$ on objects and by $f \mapsto X_*(f)$ on morphisms.

Bourke proves the result stated in the beginning of the section for identity type categories by showing two things:

(1) the category $X_*(0) \mathbin{/\!\!/} \mathcal{C}$ has R-globular products, and moreover U preserves globular products,

(2) the endomorphism theory $J_R : \Theta_0^{op} \to End(R)$ is contractible.

This would indeed be sufficient, as then $End(R)$ will be the desired globular theory, and the composite map

$$End(R) \xrightarrow{K_R} X_*(0) \mathbin{/\!\!/} \mathcal{C} \xrightarrow{U} \mathcal{C}$$

will be an $End(R)$-algebra structure on X_*, hence the desired weak ω-groupoid. In the current setting, we are able to prove the first statement.

Lemma 26. *The category $X_*(0) \mathbin{/\!\!/} \mathcal{C}$ has R-globular products, and U preserves globular products.*

Proof. We need to show that the diagram $R^{\bar{n}} : (\mathbb{G}^{op})^{\bar{n}} \to X_*(0) \mathbin{/\!\!/} \mathcal{C}$ has a limit for every table of dimensions $\bar{n} = (n_1, \ldots, n_k)$. We proceed by induction on the length k of the table of dimensions. For the base case $k = 1$, we have $R(\bar{n}) = r^0_{n_1}$, and the projection is the identity.

Now suppose that $R^{\bar{n}}$ has a limit $R(\bar{n}) = X_*(0) \xrightarrow{r_{\bar{n}}} X_*(\bar{n})$ with projections $\pi^{\bar{n}}_i : X_*(\bar{n}) \to X_*(n_i)$ for some odd $k \geq 1$. We extend the table of dimensions to $\bar{n}^+ := (n_1, \ldots, n_k, n_{k+1}, n_{k+2})$, so that the diagram $R^{\bar{n}}$ is extended to $R^{\bar{n}^+}$ by adding the maps

$$\begin{array}{ccc} X_*(n_k) & & X_*(n_{k+2}) \\ & {\scriptstyle t}\searrow \quad \swarrow{\scriptstyle s} & \\ & X_*(n_{k+1}) & \end{array}$$

where we have omitted the maps out of $X_*(0)$ for clarity. We define $X_*(\bar{n}^+)$ as the pullback

$$\begin{array}{ccccc} X_*(\bar{n}^+) & & \xrightarrow{\pi^{\bar{n}^+}_{k+2}} & & X_*(n_{k+2}) \\ {\scriptstyle q}\downarrow & & & & \downarrow{\scriptstyle s} \\ X_*(\bar{n}) & \xrightarrow{\pi^{\bar{n}}_k} & X_*(n_k) & \xrightarrow{t} & X_*(n_{k+1}) \end{array} \tag{27}$$

and the remaining projections as $\pi^{\bar{n}^+}_i := \pi^{\bar{n}}_i q$ for $i \leq k$ and

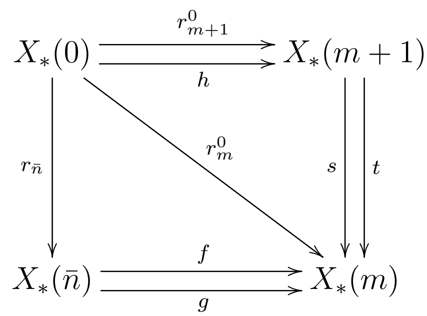
$$\pi^{\bar{n}^+}_{k+1} := t\pi^{\bar{n}}_k q = s\pi^{\bar{n}^+}_{k+2}.$$

Since $s \in \mathfrak{A}$, Proposition 12 yields there is a map $r_{\bar{n}^+} : X_*(0) \to X_*(\bar{n}^+)$, unique up to a homotopy, making $X_*(\bar{n}^+)$ a pullback in $X_*(0) /\!\!/ \mathcal{C}$. This construction verifies the universal property of the limit of $R^{\bar{n}^+}$, which follows immediately by noting that any cone on $R^{\bar{n}^+}$ is by restriction also a cone on $R^{\bar{n}}$. Thus this completes the induction.

Since R-globular products were constructed as limits in $\mathcal{C}$, the forgetful functor U preserves them. □

Note that each map $r_{\bar{n}} : X_*(0) \to X_*(\bar{n})$ is a weak equivalence (more precisely the equivalence class of maps consists of weak equivalences). This follows by a simple induction: in the base case $r_{\bar{n}} = r^0_{n_1}$ is a weak equivalence; while in the inductive step $r_{\bar{n}^+} = \tilde{q} r_{\bar{n}}$, where $\tilde{q}$ is the section of q in (27) and hence a weak equivalence, and $r_{\bar{n}}$ is a weak equivalence by induction hypothesis.

To show that the analogue of J_R is contractible (statement (2)), Bourke uses Lemma 22. This is where we face a problem induced by the fact that in the construction of X_* we use the *fiberwise* path objects. We would have to show that for all $m \in \mathbb{N}$, any parallel pair of m-cells in R whose domain is a globular product has a lifting. Let f, g be such a pair as in the diagram below.



Since f and g are maps between objects in the homotopy coslice category, we have $fr_{\bar{n}} \simeq r^0_m \simeq gr_{\bar{n}}$. In order to conclude we would wish for a homotopy $h : X_*(0) \to X_*(m+1)$ such that $sh = fr_{\bar{n}}$ and $th = gr_{\bar{n}}$. This, however, does not follow from the given conditions (apart from the case $m = 0$), as the maps $fr_{\bar{n}}$ and $fr_{\bar{n}}$ are merely homotopic, not fiberwise homotopic.

This impasse does not imply, of course, that J_R is not contractible, but merely means we cannot copy Bourke's proof strategy directly. Given this difficulty, however, it is worthwhile investigating other ways of factoring X_*. One such option is to replace $X_*(0) \mathbin{/\!\!/} \mathcal{C}$ with the fiberwise homotopy coslice category $X_*(0) \mathbin{/\!\!/}^f \mathcal{C}$ from Definition 10. This, however, leads to complications with both globular products and contractibility.

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