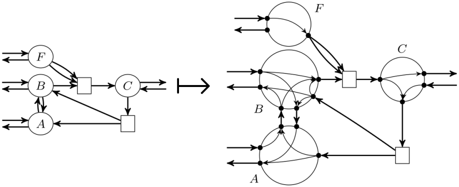
Layered Monoidal Theories

Leo Lobski (jww Fabio Zanasi)

University College London

9th International Conference on Applied Category Theory
Tallinn, 8 July 2026

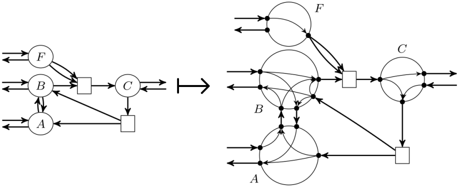
Motivation: layers of abstraction



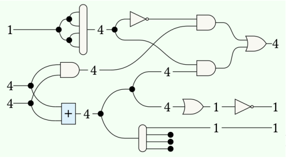
[1]

[1] Andersen, Flamm, Merkle, Stadler. *Chemical Transformation Motifs*. 2017

Motivation: layers of abstraction



[1]



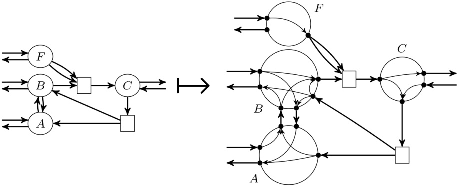
[2]

---

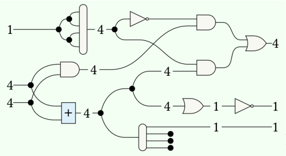
[1] Andersen, Flamm, Merkle, Stadler. *Chemical Transformation Motifs*. 2017

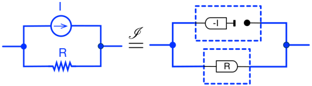
[2] Kaye. *Foundations of Digital Circuits*. 2024

Motivation: layers of abstraction



1


2



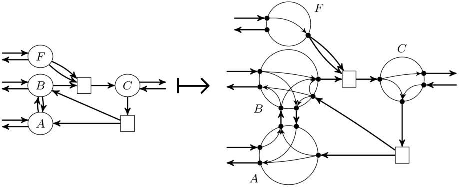
3

[1] Andersen, Flamm, Merkle, Stadler. *Chemical Transformation Motifs*. 2017

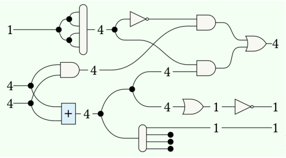
[2] Kaye. *Foundations of Digital Circuits*. 2024

[3] Boisseau, Sobociński. *String Diagrammatic Electrical Circuit Theory*. 2021

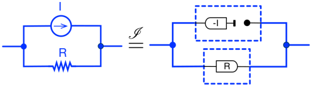
Motivation: layers of abstraction



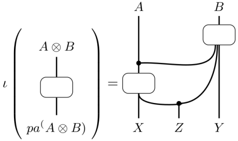
1



2



3



4

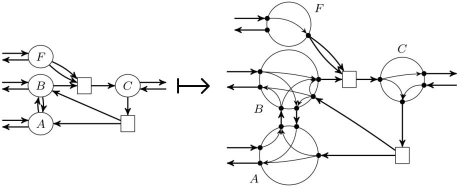
[1] Andersen, Flamm, Merkle, Stadler. *Chemical Transformation Motifs*. 2017

[2] Kaye. *Foundations of Digital Circuits*. 2024

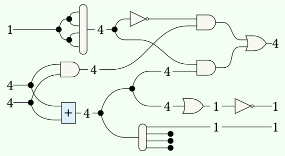
[3] Boisseau, Sobociński. *String Diagrammatic Electrical Circuit Theory*. 2021

[4] Englberger, Singh. *Causal Abstractions, Categorically Unified*. 2025

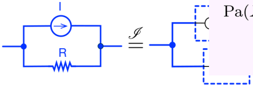
Motivation: layers of abstraction

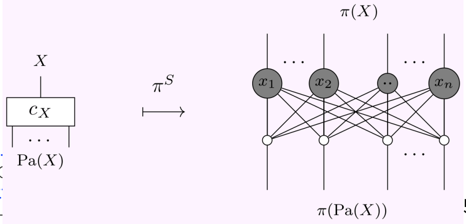


1

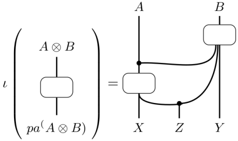


2





5



4

[1] Andersen, Flamm, Merkle, Stadler. *Chemical Transformation Motifs*. 2017

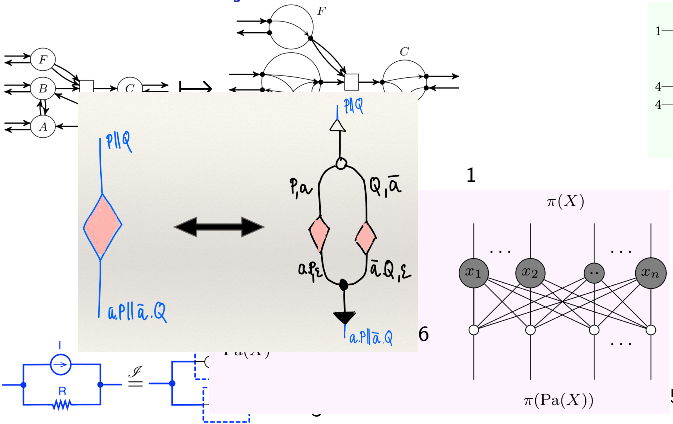
[2] Kaye. *Foundations of Digital Circuits*. 2024

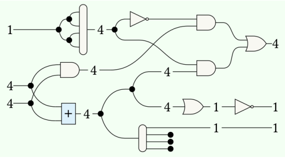
[3] Boisseau, Sobociński. *String Diagrammatic Electrical Circuit Theory*. 2021

[4] Englberger, Singh. *Causal Abstractions, Categorically Unified*. 2025

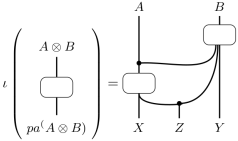
[5] Lorenz, Tull. *Causal and Compositional Abstraction*. 2026

Motivation: layers of abstraction





2

5


4

---

[1] Andersen, Flamm, Merkle, Stadler. *Chemical Transformation Motifs*. 2017

[2] Kaye. *Foundations of Digital Circuits*. 2024

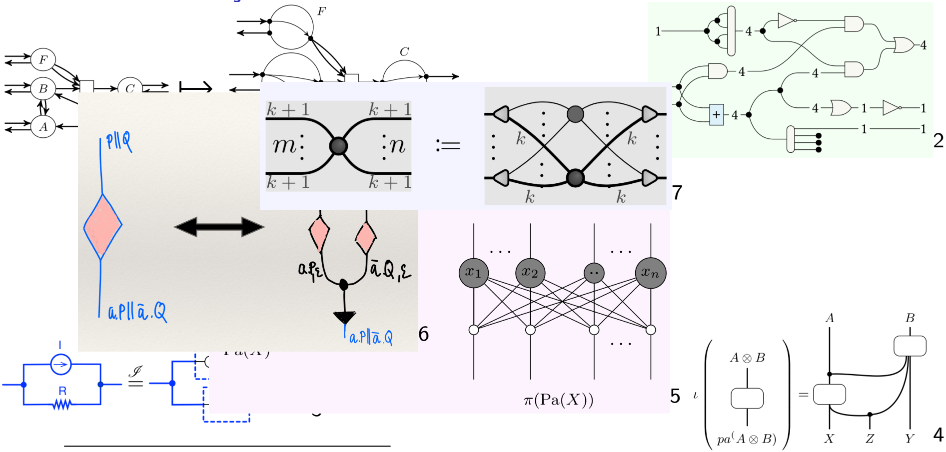
[3] Boisseau, Sobociński. *String Diagrammatic Electrical Circuit Theory*. 2021

[4] Englberger, Singh. *Causal Abstractions, Categorically Unified*. 2025

[5] Lorenz, Tull. *Causal and Compositional Abstraction*. 2026

[6] Krivine. *Diagrammatic reasoning for physical systems and composite explanations*. 2019

Motivation: layers of abstraction



[1] Andersen, Flamm, Merkle, Stadler. *Chemical Transformation Motifs*. 2017

[2] Kaye. *Foundations of Digital Circuits*. 2024

[3] Boisseau, Sobociński. *String Diagrammatic Electrical Circuit Theory*. 2021

[4] Englberger, Singh. *Causal Abstractions, Categorically Unified*. 2025

[5] Lorenz, Tull. *Causal and Compositional Abstraction*. 2026

[6] Krivine. *Diagrammatic reasoning for physical systems and composite explanations*. 2019

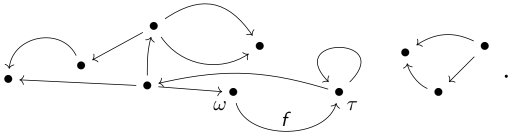
[7] Booth, Carette, Comfort. *Graphical Symplectic Algebra*. 2024

What is the purpose?

Wish to have a *syntactic* presentation for (strict) monoidal functors – akin to monoidal theories:

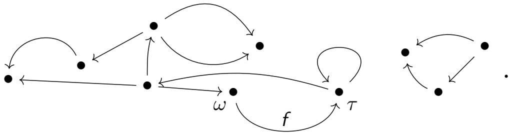
What is the purpose?

Wish to have a *syntactic* presentation for (strict) monoidal functors – akin to monoidal theories:



What is the purpose?

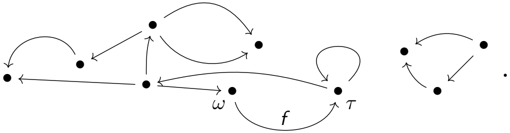
Wish to have a *syntactic* presentation for (strict) monoidal functors – akin to monoidal theories:



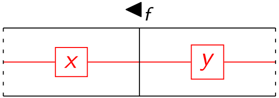
A typical term looks like:

What is the purpose?

Wish to have a *syntactic* presentation for (strict) monoidal functors – akin to monoidal theories:

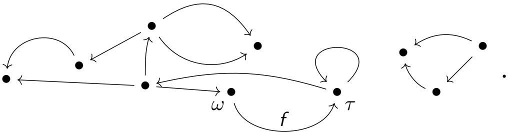


A typical term looks like:

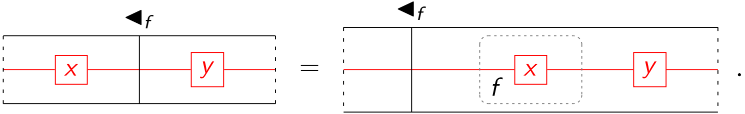


What is the purpose?

Wish to have a *syntactic* presentation for (strict) monoidal functors – akin to monoidal theories:

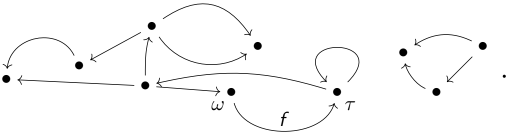


A typical term looks like:

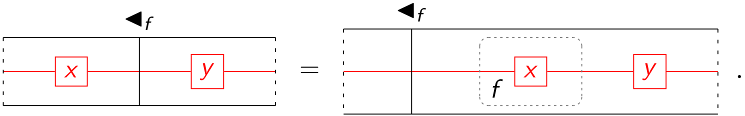


What is the purpose?

Wish to have a *syntactic* presentation for (strict) monoidal functors – akin to monoidal theories:



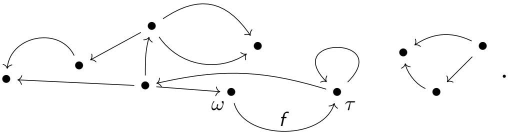
A typical term looks like:



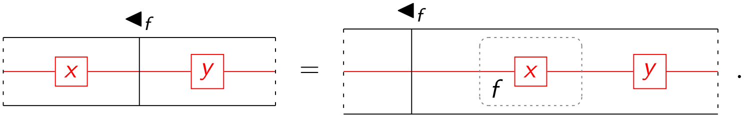
Add the *functor boundaries* in the opposite direction to allow for *partial translations*:

What is the purpose?

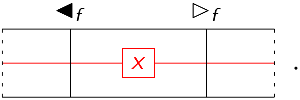
Wish to have a *syntactic* presentation for (strict) monoidal functors – akin to monoidal theories:



A typical term looks like:



Add the *functor boundaries* in the opposite direction to allow for *partial translations*:

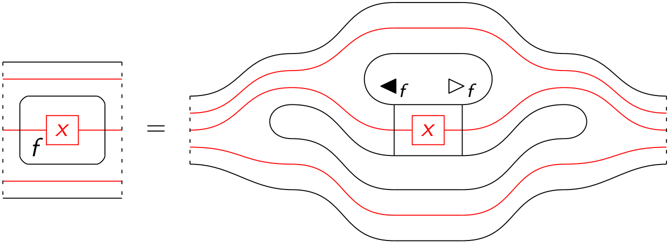


Functor coboxes

Bidirectional functor boundaries allow defining the *functor cobox*:

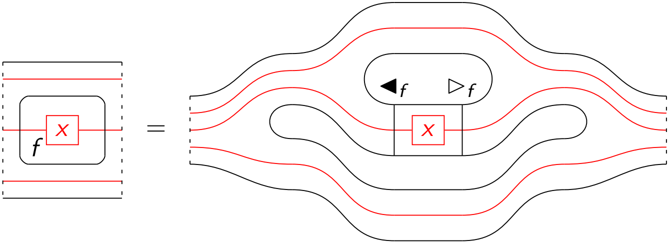
Functor coboxes

Bidirectional functor boundaries allow defining the *functor cobox*:



Functor coboxes

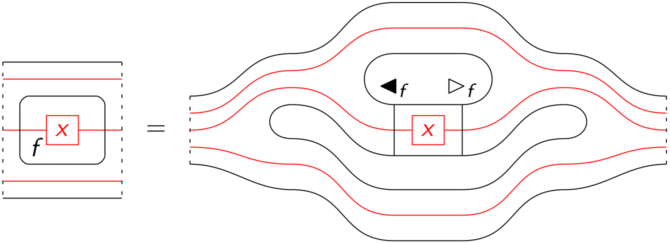
Bidirectional functor boundaries allow defining the *functor cobox*:



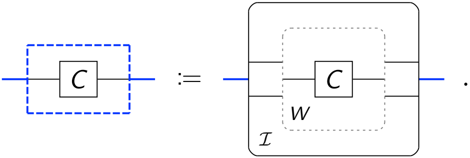
This recovers e.g. the *impedance box* used as syntactic sugar in *String Diagrammatic Electrical Circuit Theory*:

Functor coboxes

Bidirectional functor boundaries allow defining the *functor cobox*:



This recovers e.g. the *impedance box* used as syntactic sugar in *String Diagrammatic Electrical Circuit Theory*:



Layered signatures

Definition

A *layered signature* is a tuple $(\Omega, \mathcal{F}, \mathcal{M}_\omega)$, where

Layered signatures

Definition

A *layered signature* is a tuple $(\Omega, \mathcal{F}, \mathcal{M}_\omega)$, where

- Ω is a set,

Layered signatures

Definition

A *layered signature* is a tuple $(\Omega, \mathcal{F}, \mathcal{M}_\omega)$, where

- Ω is a set,
- $\mathcal{F} : \Omega \times \Omega \to \mathbf{Set}$ is a function,

Layered signatures

Definition

A *layered signature* is a tuple $(\Omega, \mathcal{F}, \mathcal{M}_\omega)$, where

- Ω is a set,
- $\mathcal{F} : \Omega \times \Omega \to \mathbf{Set}$ is a function,
- $\mathcal{M}_\omega = (C_\omega, \Sigma_\omega)$ is a monoidal signature for each $\omega \in \Omega$.

That is,

Layered signatures

Definition

A *layered signature* is a tuple $(\Omega, \mathcal{F}, \mathcal{M}_\omega)$, where

- Ω is a set,
- $\mathcal{F} : \Omega \times \Omega \to \mathbf{Set}$ is a function,
- $\mathcal{M}_\omega = (C_\omega, \Sigma_\omega)$ is a monoidal signature for each $\omega \in \Omega$.

That is,

- C_ω is a set of *colours*,

Layered signatures

Definition

A *layered signature* is a tuple $(\Omega, \mathcal{F}, \mathcal{M}_\omega)$, where

- Ω is a set,
- $\mathcal{F} : \Omega \times \Omega \to \mathbf{Set}$ is a function,
- $\mathcal{M}_\omega = (C_\omega, \Sigma_\omega)$ is a monoidal signature for each $\omega \in \Omega$.

That is,

- C_ω is a set of *colours*,
- $\Sigma_\omega : C_\omega^* \times C_\omega^* \to \mathbf{Set}$ is a function giving the *monoidal generators*.

Digital circuits: Signature

Define the layered signature $(\mathbb{N}_+, \mathcal{F}, \Sigma_n)$ as follows:

Digital circuits: Signature

Define the layered signature $(\mathbb{N}_+, \mathcal{F}, \Sigma_n)$ as follows:

- for each $n \in \mathbb{N}_+$, the set $\mathcal{F}(n, 1)$ has exactly one element,

Digital circuits: Signature

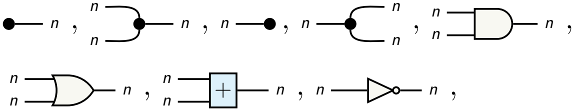
Define the layered signature $(\mathbb{N}_+, \mathcal{F}, \Sigma_n)$ as follows:

- for each $n \in \mathbb{N}_+$, the set $\mathcal{F}(n, 1)$ has exactly one element,
- for $n \neq 1$, the monoidal signature Σ_n has a single colour n, and the monoidal generators:

Digital circuits: Signature

Define the layered signature $(\mathbb{N}_+, \mathcal{F}, \Sigma_n)$ as follows:

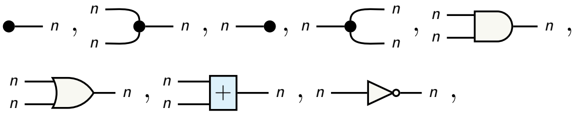
- for each $n \in \mathbb{N}_+$, the set $\mathcal{F}(n, 1)$ has exactly one element,
- for $n \neq 1$, the monoidal signature Σ_n has a single colour n, and the monoidal generators:

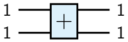


Digital circuits: Signature

Define the layered signature $(\mathbb{N}_+, \mathcal{F}, \Sigma_n)$ as follows:

- for each $n \in \mathbb{N}_+$, the set $\mathcal{F}(n, 1)$ has exactly one element,
- for $n \neq 1$, the monoidal signature Σ_n has a single colour n, and the monoidal generators:



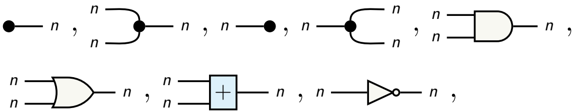
- for $n = 1$, the monoidal signature Σ_1 is as above, except that we replace the adder with 

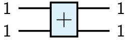
.

Digital circuits: Signature

Define the layered signature $(\mathbb{N}_+, \mathcal{F}, \Sigma_n)$ as follows:

- for each $n \in \mathbb{N}_+$, the set $\mathcal{F}(n, 1)$ has exactly one element,
- for $n \neq 1$, the monoidal signature Σ_n has a single colour n, and the monoidal generators:



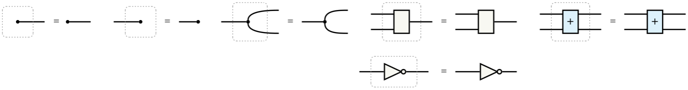
- for $n = 1$, the monoidal signature Σ_1 is as above, except that we replace the adder with 

.

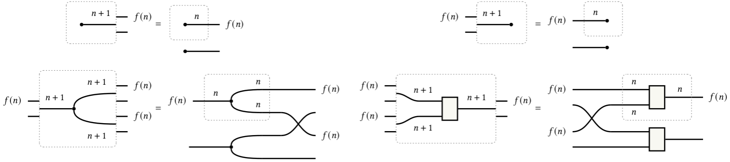
This signature is called *simple arithmetic circuits*.

Digital circuits: Theory

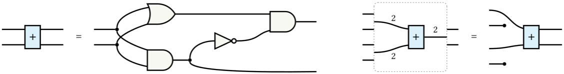
$f \in \mathcal{F}(1,1): \quad f(1) = 1$



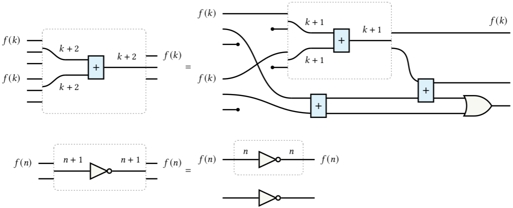
$f \in \mathcal{F}(n+1,1): \quad f(n+1) = f(n)1$



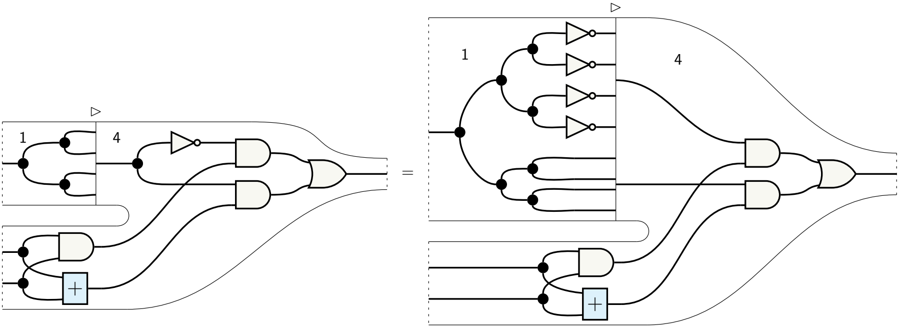
$n = 1:$



$n = k + 1$ for $k \geq 1:$



Digital circuits: A simple arithmetic logic unit



Semantics

theory	semantics (roughly)
monoidal theory	monoidal category
“forward” boundaries	monoid in the category of opfibrations
“backward” boundaries	comonoid in the category of fibrations
bidirectional boundaries	collage of certain diagram of profunctros

Related work

- Internal string diagrams for TQFTs and profunctors:

Related work

- Internal string diagrams for TQFTs and profunctors:
  - Bruce Bartlett, Christopher L. Douglas, Christopher J. Schommer-Pries, and Jamie Vicary. 2015. *Modular categories as representations of the 3-dimensional bordism 2-category*.
  - Nick Hu and Jamie Vicary. 2021. *Traced Monoidal Categories as Algebraic Structures in Prof*.

Related work

- Internal string diagrams for TQFTs and profunctors:
  - Bruce Bartlett, Christopher L. Douglas, Christopher J. Schommer-Pries, and Jamie Vicary. 2015. *Modular categories as representations of the 3-dimensional bordism 2-category*.
  - Nick Hu and Jamie Vicary. 2021. *Traced Monoidal Categories as Algebraic Structures in Prof*.
- Open diagrams for coend calculus and string diagram collages:

Related work

- Internal string diagrams for TQFTs and profunctors:
  - Bruce Bartlett, Christopher L. Douglas, Christopher J. Schommer-Pries, and Jamie Vicary. 2015. *Modular categories as representations of the 3-dimensional bordism 2-category*.
  - Nick Hu and Jamie Vicary. 2021. *Traced Monoidal Categories as Algebraic Structures in Prof*.
- Open diagrams for coend calculus and string diagram collages:
  - Mario Román. 2021. *Open Diagrams via Coend Calculus*.
  - Dylan Braithwaite and Mario Román. 2023. *Collages of String Diagrams*.

Related work

- Internal string diagrams for TQFTs and profunctors:
  - Bruce Bartlett, Christopher L. Douglas, Christopher J. Schommer-Pries, and Jamie Vicary. 2015. *Modular categories as representations of the 3-dimensional bordism 2-category*.
  - Nick Hu and Jamie Vicary. 2021. *Traced Monoidal Categories as Algebraic Structures in Prof*.
- Open diagrams for coend calculus and string diagram collages:
  - Mario Román. 2021. *Open Diagrams via Coend Calculus*.
  - Dylan Braithwaite and Mario Román. 2023. *Collages of String Diagrams*.
- Holey string diagrams:

Related work

- Internal string diagrams for TQFTs and profunctors:
  - Bruce Bartlett, Christopher L. Douglas, Christopher J. Schommer-Pries, and Jamie Vicary. 2015. *Modular categories as representations of the 3-dimensional bordism 2-category*.
  - Nick Hu and Jamie Vicary. 2021. *Traced Monoidal Categories as Algebraic Structures in Prof*.
- Open diagrams for coend calculus and string diagram collages:
  - Mario Román. 2021. *Open Diagrams via Coend Calculus*.
  - Dylan Braithwaite and Mario Román. 2023. *Collages of String Diagrams*.
- Holey string diagrams:
  - Matt Earnshaw and Mario Román. 2024. *Context-Free Languages of String Diagrams*.
  - James Hefford and Cole Comfort. 2023. *Coend Optics for Quantum Combs*.

Related work

- Internal string diagrams for TQFTs and profunctors:
  - Bruce Bartlett, Christopher L. Douglas, Christopher J. Schommer-Pries, and Jamie Vicary. 2015. *Modular categories as representations of the 3-dimensional bordism 2-category*.
  - Nick Hu and Jamie Vicary. 2021. *Traced Monoidal Categories as Algebraic Structures in Prof*.
- Open diagrams for coend calculus and string diagram collages:
  - Mario Román. 2021. *Open Diagrams via Coend Calculus*.
  - Dylan Braithwaite and Mario Román. 2023. *Collages of String Diagrams*.
- Holey string diagrams:
  - Matt Earnshaw and Mario Román. 2024. *Context-Free Languages of String Diagrams*.
  - James Hefford and Cole Comfort. 2023. *Coend Optics for Quantum Combs*.
- Semantics:

Related work

- Internal string diagrams for TQFTs and profunctors:
  - Bruce Bartlett, Christopher L. Douglas, Christopher J. Schommer-Pries, and Jamie Vicary. 2015. *Modular categories as representations of the 3-dimensional bordism 2-category*.
  - Nick Hu and Jamie Vicary. 2021. *Traced Monoidal Categories as Algebraic Structures in Prof*.
- Open diagrams for coend calculus and string diagram collages:
  - Mario Román. 2021. *Open Diagrams via Coend Calculus*.
  - Dylan Braithwaite and Mario Román. 2023. *Collages of String Diagrams*.
- Holey string diagrams:
  - Matt Earnshaw and Mario Román. 2024. *Context-Free Languages of String Diagrams*.
  - James Hefford and Cole Comfort. 2023. *Coend Optics for Quantum Combs*.
- Semantics:
  - Joe Moeller and Christina Vasilakopoulou. 2020. *Monoidal Grothendieck construction*.

References

- L. *Layered monoidal theories*. 2025. PhD thesis.
- L. and Fabio Zanasi. 2026. *Layered monoidal theories I: Diagrammatic algebra and applications*
- L. and Fabio Zanasi. 2026. *Layered monoidal theories II: Fibrational semantics*

Thank you for your attention!