A note on sheaves and étale bundles

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These notes are meant to accompany a talk for the Pirate seminar organised by students in the MSc Mathematics degree at the University of Amsterdam in the spring of 2019. In the spirit of modern-day piracy, the notes are more or less directly extracted from my BSc project [1], which, albeit long and painfully detailed, hardly contains any novel mathematics. Consequently, even less originality and insight is to be expected here. The main reference for the relevant part of the project is Mac Lane and Moerdijk [2].

1 Presheaves and sheaves

In its full generality, an $\mathcal{S}$*-valued presheaf* on a category $\mathcal{C}$ is a functor,

$$P : \mathcal{C}^{op} \to \mathcal{S}.$$

In this note, however, we will be only considering **Set**-valued sheaves and presheaves over a topological space X. More precisely, given a topological space X, let $\mathcal{O}(X)$ denote the poset category of open subsets of X. A presheaf is then a functor

$$P : \mathcal{O}(X)^{op} \to \mathbf{Set}.$$

An element $s \in P(V)$ is called a *section* of P over $V \subseteq X$. Note that whenever we have $U, V \in \mathcal{O}(X)$ with $U \subseteq V$, we have a map $P(V) \to P(U)$ in **Set**. We will denote the image of a section $s \in P(V)$ under this map $s|_U$ and call it the *restriction* of s to U.

Definition 1. *(Sheaf)* A presheaf $P : \mathcal{O}(X)^{op} \to \mathbf{Set}$ is a *sheaf* if for any family $(U_i)_{i\in I}$ of open subsets of X, and for any family $(s_i)_{i\in I}$ such that $s_i \in P(U_i)$ for each $i \in I$ with the property

$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j} \quad \forall i, j \in I, \tag{2}$$

there exists a unique $s \in P\left(\bigcup_i U_i\right)$ such that

$$s|_{U_i} = s_i \quad \forall i \in I.$$

Proposition 3. *A presheaf P on a topological space X is a sheaf if and only if the following diagram is an equaliser for any family of open sets $(U_i)_{i\in I}$, writing $U = \bigcup_i U_i$.*

$$P(U) \xrightarrow{\quad e \quad} \prod_i P(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} P(U_i \cap U_j) \ ,$$

where for $s \in P(U)$ and for $(s_i)_{i\in I} \in \prod_i P(U_i)$,

$$\begin{aligned} e(s) &= (s|_{U_i})_{i\in I}, \\ p((s_i)_{i\in I}) &= (s_i|_{U_i\cap U_j})_{i,j\in I}, \\ q((s_i)_{i\in I}) &= (s_j|_{U_i\cap U_j})_{i,j\in I}. \end{aligned}$$

For a topological space X, we write $\mathbf{PSh}(X)$ for the category whose objects are all presheaves on X and morphisms are natural transformations of functors. Similarly, we write $\mathbf{Sh}(X)$ for the full subcategory of $\mathbf{PSh}(X)$ of sheaves on X.

2 Germs and stalks

Let P be a presheaf over X and let x be a point in X. If U and V are some open neighbourhoods of x, we say that the sections $s \in P(U)$ and $t \in P(V)$ *have the same germ* at x if there exists an open neighbourhood W of x such that $W \subseteq U \cap V$ and $s|_W = t|_W \in P(W)$, in which case we write sG_xt.

Proposition 4. *The relation sG_xt is an equivalence relation on sections over open neighbourhoods of $x \in X$.*

Definition 5. *(Germ)* For a presheaf P over X, let $x \in X$ and U an open neighbourhood of x. For a section $s \in P(U)$, the *germ* of s at x is the equivalence class of s under the relation G_x.

Following the notation in Mac Lane and Moerdijk [2, p. 83], we denote the germ of s at x by $\mathrm{germ}_x s$.

Definition 6. *(Stalk)* Given a presheaf P on X, let the *stalk* of P at x, denoted by P_x, be the set of all germs at x. That is,

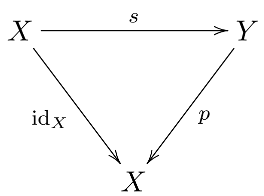
$$P_x = \{\mathrm{germ}_x s : s \in P(U), \text{ where } U \subseteq X \text{ open and } x \in U\}.$$

3 Bundles and sections

Definition 7. *(Bundle)* A *bundle* over a topological space X is an object in the slice category $\mathbf{Top}/X$.

We denote the category of bundles over X by $\mathbf{Bund}(X)$, where the morphisms are simply the morphisms in $\mathbf{Top}/X$, hence $\mathbf{Bund}(X) \coloneqq \mathbf{Top}/X$.

Definition 8. *(Section)* A *(global) section* of a bundle $p : Y \to X$ is a morphism s from id_X to p in $\mathbf{Bund}(X)$, as in the diagram.



That is, a section is a continuous function $s : X \to Y$ with $ps = \mathrm{id}_X$.

It is useful to allow sections to be defined on some open subset of X only. In detail, if $U \subseteq X$ is open, the bundle $p : Y \to X$ restricts to a bundle

$$p_U : p^{-1}(U) \to U.$$

A section of p_U (or a section of p over U) is then a morphism s from $i : U \hookrightarrow X$ (inclusion) to p in $\mathbf{Bund}(X)$, that is, a continuous function $s : U \to Y$ such that $ps = i$. A section of p_U is called *local*.

Definition 9. *(Étale bundle)* A bundle $p : Y \to X$ is *étale* if p is a *local homeomorphism*, that is, every $y \in Y$ has an open neighbourhood V such that pV is open in X and the restriction

$$p|_V : V \to pV$$

is a homeomorphism.

We denote the full subcategory of $\mathbf{Bund}(X)$ consisting of étale bundles by $\mathbf{Et}(X)$.

4 Key results

Theorem 10. *Let X be a topological space. Then there is an adjunction (left adjoint on the left)*

$$\Lambda : \mathbf{PSh}(X) \rightleftarrows \mathbf{Bund}(X) : \Gamma.$$

The functor Γ in the above theorem maps each bundle to the presheaf taking each open set $U \subseteq X$ to the set of (local) sections of p_U. This presheaf is moreover a sheaf. The functor Λ takes a presheaf P to the bundle $p : \coprod_{x \in X} P_x \to X$ (whose domain is the disjoint union of stalks of P) defined by $p(\mathrm{germ}_x s) = x$. This bundle is moreover étale.

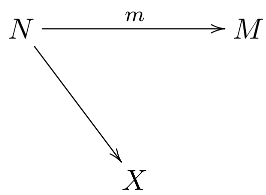
Theorem 11. *Let X be a topological space. The adjunction in Theorem 10 restricts to an equivalence of categories*

$$\mathbf{Sh}(X) \rightleftarrows \mathbf{Et}(X).$$

5 Sheaves, abstractly

The following is (part of) Definition 2.27 in van Oosten [3].

Definition 12. Let $\mathcal{E}$ be a topos with a subobject Ω, and let $j : \Omega \to \Omega$ a Lawvere-Tierney topology on $\mathcal{E}$. We say that an object X is a *sheaf* for j (or a j-sheaf) if for any dense subobject $m : N \to M$ and a partial map



there is exactly one extension to a map $M \to X$.

References

[1] Leo Lobski. *Sheaves on topological spaces and their logic.* BSc project. University of Edinburgh. Spring 2018.

[2] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in geometry and logic. A first introduction to topos theory.* Springer, 1992. ISBN: 0387977104.

[3] Jaap van Oosten. *Topos theory.* Course notes. Utrecht University. Autumn 2018.