Categorical embeddings of effect algebras

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Outline

The structure of a measurement in quantum mechanics

Effect algebras

Finite Boolean algebras are dense in effect algebras

Partitions of unity

The structure of a measurement in quantum mechanics
(theorist's perspective)

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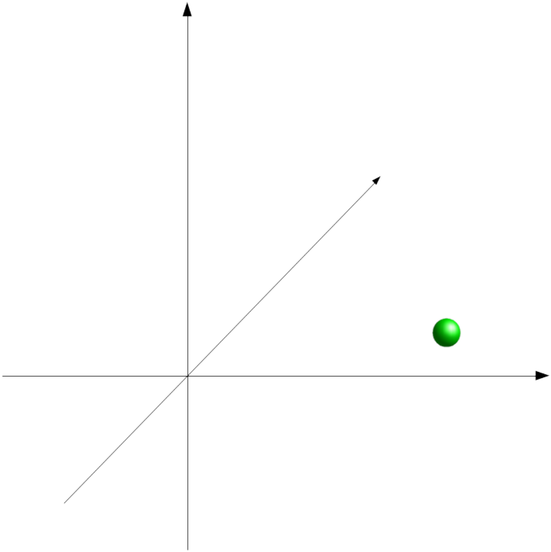
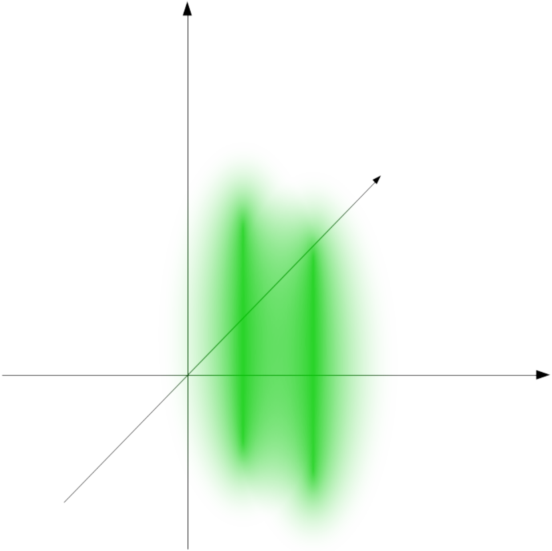


The structure of a measurement in quantum mechanics

(theorist's perspective)



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- Operational quantum mechanics: replace the Hilbert space with the set of *effects*: physical outcomes which may actually occur

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(E2) if $a \perp b$ and $(a \oplus b) \perp c$, then $b \perp c$ and $a \perp (b \oplus c)$ as well as

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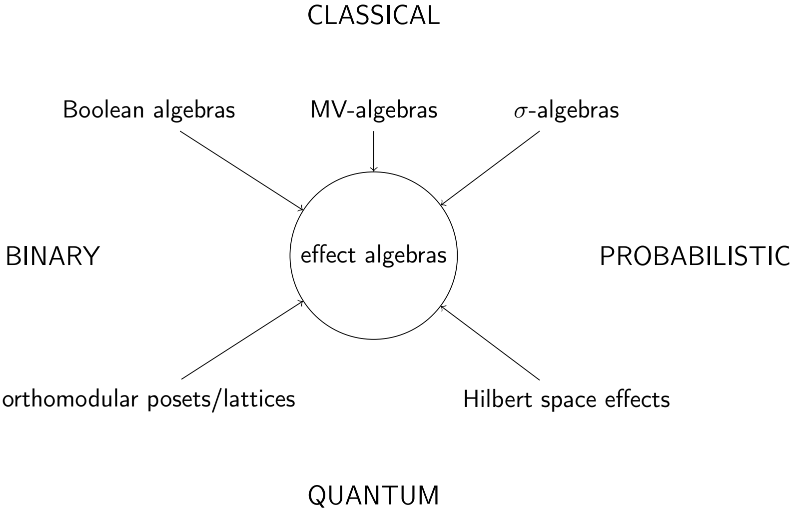
(E4) if $a \perp 1$, then $a = 0$.

Examples


CLASSICAL
Boolean algebras
MV-algebras
σ-algebras
BINARY
effect algebras
PROBABILISTIC
orthomodular posets/lattices
Hilbert space effects
QUANTUM

Goal for today

FinBA is dense in **EAlg**

Dense subcategories

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We say that C is a *canonical colimit of $\mathcal{A}$-objects* if the canonical diagram has a colimit with vertex C and coprojections

$$D\left(A \xrightarrow{f} C\right) \xrightarrow{f} C,$$

where $f : A \to C$ ranges through the objects of $\mathcal{A}/C$.

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Definition

A small, full subcategory $\mathcal{A}$ of a category $\mathcal{C}$ is *dense* if every object of $\mathcal{C}$ is a canonical colimit of $\mathcal{A}$-objects.

The nerve functor

Definition

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Proposition

Let $\mathcal{A}$ be a small, full subcategory of a category $\mathcal{C}$. Then $\mathcal{A}$ is dense if and only if the nerve functor $N_{\mathcal{A}}$ is full and faithful.

The test functor

Definition

Let E be an effect algebra and let $n \in \mathbb{N}$. An *n-test* is a list of elements of E of length n

$$(e_1, \ldots, e_n)$$

such that their sum $\bigoplus_{i=1}^{n} e_i$ exists and is equal to 1.

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We use this to define a functor for each effect algebra E:

$$\begin{aligned} T(E) : \mathbb{N} &\to \mathbf{Set} \\ n &\mapsto T(E)(n) \\ \left(n \xrightarrow{f} m\right) &\mapsto (T(E)(n) \to T(E)(m)) \\ (e_1, \dots, e_n) &\mapsto \left(\bigoplus_{i \in f^{-1}(j)} e_i \right)_{j=1,\dots,m} \end{aligned}$$

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This further lifts to the *test functor*:

$$\begin{aligned} T : \mathbf{EAlg} &\to [\mathbb{N}, \mathbf{Set}] \\ E &\mapsto T(E) \\ (\alpha : E \to F) &\mapsto T(\alpha), \end{aligned}$$

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$$\begin{aligned} T(\alpha)_n : T(E)(n) &\to T(F)(n) \\ (e_1, \ldots, e_n) &\mapsto (\alpha(e_1), \ldots, \alpha(e_n)). \end{aligned}$$

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Theorem (Staton and Uijlen 2015)

The test functor $T : \mathbf{EAlg} \to [\mathbb{N}, \mathbf{Set}]$ *is full and faithful.*

The test is the nerve (up to...)

We have an equivalence of categories:

$$- \circ \mathcal{P}^{op} : [\mathbf{FinBA}^{op}, \mathbf{Set}] \to [\mathbb{N}, \mathbf{Set}].$$

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The test functor $T : \mathbf{EAlg} \to [\mathbb{N}, \mathbf{Set}]$ *is naturally isomorphic to the nerve functor composed with the above equivalence:*

$$N_{\mathbf{FinBA}}(-) \circ \mathcal{P}^{op} : \mathbf{EAlg} \to [\mathbb{N}, \mathbf{Set}].$$

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Corollary

The category **FinBA** *is a dense subcategory of* **EAlg**.

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Definition

Let E be an effect algebra. A multiset (A, η) such that $A \subseteq E$ is a *partition of unity* if it is summable, $0 \notin A$, and

$$\bigoplus_{a \in A} \eta(a) \cdot a = 1.$$

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  - $\mathcal{P} \leq \mathcal{Q}$ if $\mathcal{P}$ can be partitioned into $|\mathcal{Q}|$ parts such that the sum of each such part is a unique (up to the multiplicity) element of $\mathcal{Q}$.
- Partitions of unity are in one-to-one correspondence with images of discrete positive operator valued measures (POVMs).
  - The refinement order corresponds to coarse-graining.

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Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, and let $\mathfrak{C}$ be an isomorphism-closed subclass of objects of $\mathcal{C}$. We say that F is *essentially injective on* $\mathfrak{C}$*-objects* if for any objects $C, B \in \mathfrak{C}$, having $F(C) \simeq F(B)$ implies $C \simeq B$.

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Conjecture

The functor

$$\mathrm{Part} : \mathbf{EAlg} \to \mathbf{Pos}$$

is essentially injective on effect algebras which do not have minimal partitions of unity of cardinality 2 *or less.*

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- As a byproduct, we have formulated Bohr's doctrine in terms of effect algebras and category theory.
- Open problem 1: Characterise those functors $[\mathbb{N}, \mathbf{Set}]$ which correspond to an effect algebra.
- Open problem 2: Show that not just tests but also partitions of unity have enough information to reconstruct an effect algebra.

References

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Thank you for your attention!