Partition functor for effect algebras: a model of quantum measurements

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Outline

The structure of a measurement in quantum mechanics

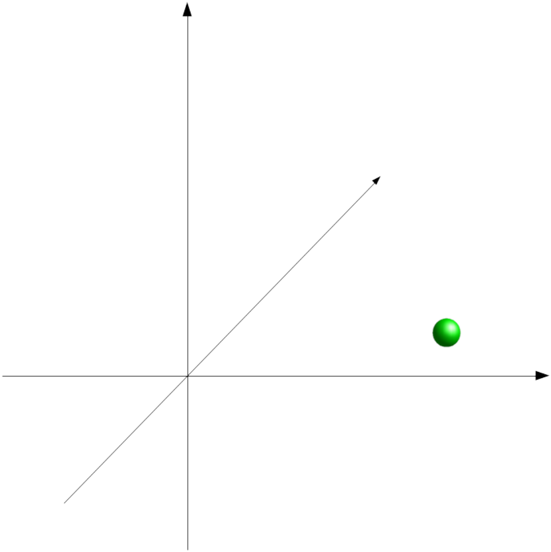
Effect algebras

Partitions of unity

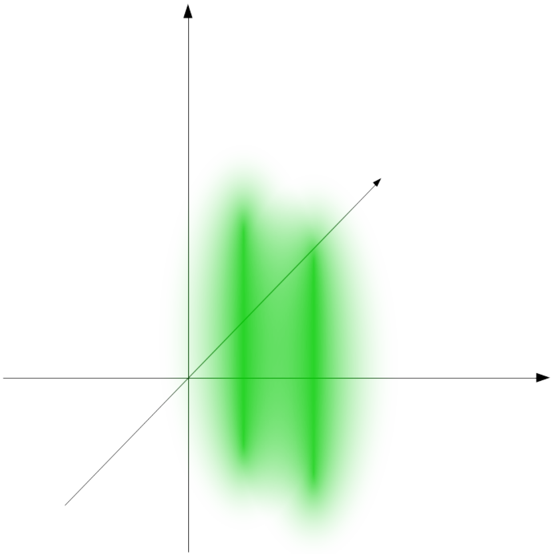
Boolean algebras and orthoalgebras

Finite MV-algebras

The structure of a measurement in quantum mechanics
(theorist's perspective)



The structure of a measurement in quantum mechanics
(theorist's perspective)



The structure of a measurement in quantum mechanics

(theorist's perspective)

- A system is modelled by a Hilbert space
- We have access to the system via measurements only
  - A finite *positive operator valued measure* is a finite set $\{A_i\}_{i \in I}$ of positive semi-definite self-adjoint operators such that
$$\sum_{i \in I} A_i = I.$$
  - Philosophical problem: is the information contained in the measurements sufficient to know the system?
- Operational quantum mechanics: replace the Hilbert space with the set of *effects*: physical outcomes which may actually occur

Effect algebras

Definition

An *effect algebra* is a partial algebra $(E, 0, 1,', \perp, \oplus)$ such that the following hold for all $a, b, c \in E$:

(E1) if $a \perp b$, then $b \perp a$ and $a \oplus b = b \oplus a$,

(E2) if $a \perp b$ and $(a \oplus b) \perp c$, then $b \perp c$ and $a \perp (b \oplus c)$ as well as

$$(a \oplus b) \oplus c = a \oplus (b \oplus c),$$

(E3) $a \perp a'$ and $a \oplus a' = 1$, and if $a \perp b$ such that $a \oplus b = 1$, then $b = a'$,

(E4) if $a \perp 1$, then $a = 0$.

Examples of effect algebras

- Boolean algebras: $a \perp b$ iff $a \wedge b = 0$ and $a \oplus b = a \vee b$.
- Orthoalgebras: replace (E4) with the 'sharpness condition'.
- MV-algebras: $a \perp b$ iff $a \leq b'$ and $a \oplus b = a + b$.
- Hilbert space effects $\mathcal{E}(H)$: positive operators between 0 and 1 with truncated addition.
  - Assumption: the (partial) algebraic structure of $\mathcal{E}(H)$ has all the physically relevant information.

Partitions of unity

Let E be an effect algebra. A multiset (A, η) such that $A \subseteq E$ is *summable* if the sum

$$\bigoplus_{a \in A} \eta(a) \cdot a$$

exists (if it exists it is well-defined).

Definition

Let E be an effect algebra. A multiset (A, η) such that $A \subseteq E$ is a *partition of unity* if it is summable, $0 \notin A$, and

$$\bigoplus_{a \in A} \eta(a) \cdot a = 1.$$

Partitions of unity

- Part(E) is partially ordered "by refinement":
  - $\mathcal{P} \leq \mathcal{Q}$ if $\mathcal{P}$ can be partitioned into $|\mathcal{Q}|$ parts such that the sum of each such part is a unique (up to the multiplicity) element of $\mathcal{Q}$.
- Partitions of unity are in one-to-one correspondence with images of discrete positive operator valued measures (POVMs).
  - The refinement order corresponds to coarse-graining.

The partitions of unity functor

- Partitions of unity extend to a functor $\mathrm{Part} : \mathbf{EAlg} \to \mathbf{Pos}$.

Definition

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, and let $\mathfrak{C}$ be an isomorphism-closed subclass of objects of $\mathcal{C}$. We say that F is *essentially injective on* $\mathfrak{C}$*-objects* if for any objects $C, B \in \mathfrak{C}$, having $F(C) \simeq F(B)$ implies $C \simeq B$.

Conjecture

The functor

$$\mathrm{Part} : \mathbf{EAlg} \to \mathbf{Pos}$$

is essentially injective on effect algebras which do not have minimal partitions of unity of cardinality 2 *or less.*

Boolean algebras and orthoalgebras

Theorem
The functor

$$\mathrm{FinSub} : \mathbf{BAlg} \to \mathbf{Pos}$$

is essentially injective on Boolean algebras with more than four elements.

- Finite subalgebra poset of a Boolean algebra is dually isomorphic to its partitions of unity poset.
- Proved for the poset of all Boolean algebras by Sachs (1961) and independently by Filippov (1965). Grätzer, Koh and Makkai (1972) gave an alternative proof.

Theorem (Harding, Heunen, Lindenhovius and Navara, 2019)
If A is a proper orthoalgebra, then $\mathrm{BSub}(A)$ *has enough directions and* $\mathrm{Dir}(\mathrm{BSub}(A))$ *is an orthoalgebra isomorphic to A.*

MV-algebras

Definition

An *MV-algebra* is an algebra $(M, 0, 1,', +)$ such that the following axioms hold:

(MV1) $a + (b + c) = (a + b) + c$, *(associativity)*

(MV2) $a + b = b + a$, *(commutativity)*

(MV3) $a + 0 = a$,

(MV4) $a + 1 = 1$,

(MV5) $a'' = a$,

(MV6) $0' = 1$,

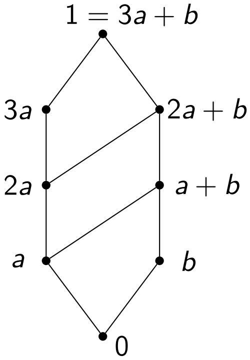
(MV7) $a + a' = 1$,

(MV8) $(a' + b)' + b = (a + b')' + a$.

Examples

- Boolean algebras
- The unit interval

Example



Partitions determine finite MV-algebras

Theorem (L. 2020)

The functor

$$\mathrm{Part} : \mathbf{FinMV} \to \mathbf{Pos}$$

is essentially injective on algebras with more than four elements.

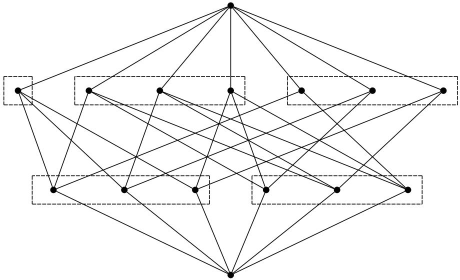
Setoid quotients

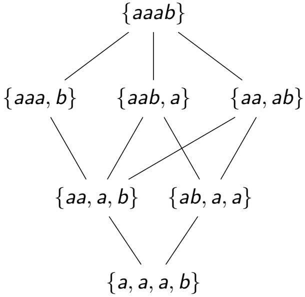
Definition

Let $(X, \sim)$ be a finite setoid. Its *setoid quotient* is the quotient poset $\mathrm{PartM}(X)/\sim$.

Lemma

Every poset of partitions of unity of a finite MV-algebra is isomorphic to a setoid quotient (and conversely).





Proof outline

- View partition posets as setoid quotients.
- Posets have the same height, hence setoids have the same cardinality.
- Count the number of atoms. Obtain that the setoids have the same number of equivalence classes, and the same number of equivalence classes whose cardinality is exactly one.

Lemma

Let $(X, \sim)$ be a finite setoid with more than two elements. There is only one $\sim$-equivalence class (namely, X) if and only if the setoid quotient $\mathrm{SQuot}(X, \sim)$ *has the second least element.*

Proof outline

Lemma

Let $(X, \sim)$ be a finite setoid and let E be some $\sim$-equivalence class. Let us denote the partition of X containing E and no other non-singleton sets (or only the singleton sets if E is a singleton) by

$$P_E \coloneqq \{E\} \cup \{\{x\} : x \in X \setminus E\}.$$

Then

$$\downarrow [P_E] \simeq \mathsf{SQuot}(E, \sim|_E),$$

where the downset on the left-hand side is taken in $\mathsf{SQuot}(X, \sim)$ *and $\sim|_E$ is the restriction of $\sim$ to E, in other words, the total equivalence relation on E.*

- There is the same number of equivalence classes of cardinality at least four.
- Count the number of remaining equivalence classes and their cardinalities.

Conclusion

- We have formulated Bohr's doctrine in terms of effect algebras and category theory.
- The result for Boolean algebras shows that classical measurements are informationally complete.
- The result for MV-algebras can have significance for the theory of MV-algebras, setoids and multisets.

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Thank you for your attention!